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THE ABELIAN EQUATIONS OF THE TENTH DEGREE IRREDUCIBLE IN A GIVEN DOMAIN OF RATIONALITY

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CHARLES GUSTAVE PAUL KUSCHKE.

CONTENTS.

INTRODUCTION.

In the following pages is found the number of types of Abelian equations of the tenth degree irreducible in a given domain of rationality. The various types are discussed with respect to their construction and the character of their roots. The rational domain does not need to be the natural domain; in fact it will appear that there is no irreducible Abelian equation in the natural domain of rationality R (1).

If $f(x) = 0$ is an irreducible equation of the tenth degree, then all the ten roots

x_i ($i = 1$ to 10) must be distinct, for otherwise

$$f(x) = 0 \quad \text{and} \quad \frac{d}{dx} f(x) = 0$$

would have a common root, which is impossible, as $\frac{d}{dx} f(x)$ is only of the ninth degree in x.* From the irreducibility of $f(x)$ follows further that its Galois group is transitive† and as $10 = 2 \cdot 5$ this group must be imprimitive.‡ Our first question therefore is: What are all the imprimitive groups of the tenth degree? These groups are found by Cole§ and divided into three types.

1. The groups which contain only 5 systems of intransitivity.
2. The groups which contain only 2 systems of intransitivity.
3. The groups which contain both 5 and 2 systems of intransitivity.

To type 1 belong thirteen groups, the largest of which is of order 3840 containing all the other twelve as subgroups

To type 2 belong fifteen groups. The largest group is of order 28 800 and contains all the other fourteen groups as subgroups.

Type 3 includes eight groups, of which the largest is of order 240.

Netto‖ showed that the group of an Abelian equation is an Abelian group and vice versa. Our first problem is, therefore, to search for all Abelian groups among the above thirty-six imprimitive groups. Expressing the group of order 240 in type 3 as a substitution group, we see the following relation between the eight groups of type 3, which we may call

$$G_{240}; \quad G_{120}; \quad G'_{120}; \quad G_{40}; \quad G_{20}; \quad G'_{20}; \quad G_{10}; \quad G'_{10}$$

where the subscripts indicate the order:

G_{240} contains all the remaining seven groups as subgroups.

G'_{20} is a subgroup of G'_{120} whereas G_{20} is a subgroup of G_{120} as well as of G_{40} and contains both G'_{10} and G_{10} as subgroups. Writing this relation in a better form we have the following scheme

$$G_{240}\left\{\begin{array}{l} \left.\begin{array}{l} G_{120} \\ G_{40} \end{array}\right\} G_{20}\left\{\begin{array}{l} G_{10} \\ G'_{10} \end{array}\right. \\ G'_{120}\; G'_{20} \end{array}\right.$$

As G'_{10} is the dihedral group of order 10 and G'_{20} contains five subgroups of order 4 which are conjugate by Sylow's theorem, it follows that the cyclical group G_{10} is the only Abelian group of type 3.

We shall use as G_{10} the ten powers of

$$s = (x_1 x_7 x_3 x_9 x_5 x_6 x_2 x_8 x_4 x_{10})$$

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* Weber, *Algebra*, vol. I, p. 454.

† Weber, *Algebra*, vol. I, p. 482.

‡ Netto, *Substitutionentheorie*, §237.

§ Cole, *Quart. Journ. Math.*, vol. 27, 1895, p. 39; but also Miller, *Bull. Amer. Math. Soc.*, vol. 1, pp. 67–72.

‖ *Loc. cit.*, §180, and C. Jordan, *Traité d'Analyse*, §402.

and in dealing with substitutions we shall write only the subscripts of the x's, replacing the subscript ten by zero.

Whenever we can show that a given equation $f(x) = 0$ has a Galois group with two as well as five systems of imprimitivity, but which cannot contain G'_{10} or G'_{20}, $f(x) = 0$ must be, according to the above scheme, an equation belonging to G_{10} if $f(x)$ is irreducible. For the proof of the fact that G'_{10} and G'_{20} is not a subgroup of the Galois group of $f(x) = 0$ we shall need the two substitutions

$$\sigma_1 = (16)(20)(39)(48)(57) \text{ which is in } G'_{10}$$

and

$$\sigma_2 = (1748)(2936)(50) \text{ which is in } G'_{20}$$

These three substitutions s, σ_1, σ_2 already show that we have chosen as our two systems of imprimitivity

$$[x_1x_2x_3x_4x_5], \quad [x_6x_7x_8x_9x_{10}]$$

and as our five systems of imprimitivity

$$[x_1x_6], \quad [x_2x_7], \quad [x_3x_8], \quad [x_4x_9], \quad [x_5x_{10}]$$

Looking over types 1 and 2 for Abelian groups we may easily show that there is no other Abelian group.

Regarding type 2, the order of an Abelian group of degree 10 cannot exceed the order $2 \cdot 5^2 = 50$,* so that there is only one group to be considered, namely the group of order 50, which Cole† indicates by

$$G_{50} \equiv (x_1x_2x_3x_4x_5)_5(x_6x_7x_8x_9x_{10})_5(x_1x_6 \cdot x_2x_7 \cdot x_3x_8 \cdot x_4x_9 \cdot x_5x_{10})$$

This group, however, contains the two substitutions

$$(16)(27)(38)(49)(50)$$

and

$$[(12345)(16)(27)(38)(49)(50)]^5 = (19)(74)(20)(85)(36)$$

so that G_{50} would contain more than one substitution of order 2, which is impossible here by Sylow's theorem, hence there is no Abelian group of type 2.

Regarding type 1 we note that the only bases for the construction of the thirteen groups in question are

$$(16)(27)(38)(49)(50)$$

and

$$\{(16)(27)(38)(49)(50)\}_{\text{pos}}$$

so that every group of type 1 contains the substitution

$$a = (16)(27).$$

Further, an examination of Cole's list shows that every group contains the substitution

$$b = (13579)(68024),$$

hence every group of type 1 contains

$$a^{-1}ba = (63529)(18074) \neq b,$$

and hence no group of type 1 is Abelian.

* Netto, *loc. cit.*, §170.

† *Loc. cit.*, p. 41, and Miller, *loc. cit.*

We have then the result:

All irreducible Abelian equations of the tenth degree belong to the cyclical group of order ten, which we have chosen to be

$$G_{10} \equiv \{s = (x_1x_7x_3x_9x_5x_6x_2x_8x_4x_{10})\}.$$

Throughout the whole investigation we shall assume that there is given a certain rational domain, for which the necessary and sufficient conditions for all the irreducible Abelian equations are to be derived, as the reducibility of a polynomial depends on the rational domain employed. We shall not distinguish at all between the quantities lying in the natural domain, usually denoted by $R(1)$, and quantities lying in the given domain. All quantities lying in the given rational domain shall be called "rationally known" or simply "rational quantities," for we do not restrict our investigation to the natural rational domain and have therefore to give to the expression "rational" a broader meaning than is done usually in mathematics. Whenever we mean specially a quantity lying in the natural rational domain we shall always write "quantity belonging to $R(1)$."

THE IRREDUCIBLE ABELIAN EQUATIONS OF THE TENTH DEGREE.

The corresponding group is

$$G_{10} \equiv \{s = (1739562840)\}.$$

Throughout these investigations let ϵ be a primitive root of

$$x^{10} - 1 = 0,$$

and ω be a primitive root of

$$x^5 - 1 = 0.$$

Then we may write

$$\begin{aligned}
\omega &= \epsilon^2 = -\epsilon^7 = \tfrac{1}{4}[-1 + \sqrt{5} + \sqrt{-10 - 2\sqrt{5}}],\\
\omega^2 &= \epsilon^4 = -\epsilon^9 = \tfrac{1}{4}[-1 - \sqrt{5} + \sqrt{-10 + 2\sqrt{5}}],\\
\omega^3 &= \epsilon^6 = -\epsilon = \tfrac{1}{4}[-1 - \sqrt{5} - \sqrt{-10 + 2\sqrt{5}}],\\
\omega^4 &= \epsilon^8 = -\epsilon^3 = \tfrac{1}{4}[-1 + \sqrt{5} - \sqrt{-10 - 2\sqrt{5}}],\\
\omega^5 &= \epsilon^{10} = +1,\\
&\ \epsilon^5 = -1.
\end{aligned}$$

Let us first consider the five systems of intransitivity and form the functions

$$\varphi_i = x_i + x_{i+5} \qquad (i = 1 \text{ to } 5)$$

These five functions, while formally distinct, may be numerically equal. Assuming then that at least two of these five φ_i are equal, we have by applying our group G_{10} to the relation

$$\varphi_i = \varphi_{i'} \qquad (i \neq i')$$

the result that they are all equal to one another and whenever in our equation $f(x) = 0$ we have

$$\sum_{j=1}^{j=10} x_j = 0,$$

then

$$\varphi_i = 0 \text{ and hence } x_i = -x_{i+5} \qquad (i = 1 \text{ to } 5),$$

so that our equation $f(x) = 0$ is a quintic in x^2. We note here that the transformation of $f(x) = 0$ into an equation in which

$$\sum_1^{10} x_j = 0$$

is a very simple matter and does not change the character of $f(x) = 0$. Therefore without loss of generality we may assume that in our $f(x) = 0$ the sum of the roots vanishes, as we are then gaining much simplification in our investigations.

This leads us to a division of our subject into the two parts

$$\left.\begin{array}{ll} \text{A.} & \varphi_i = 0 \\ \text{B.} & \varphi_i \neq 0 \end{array}\right\} i = 1 \text{ to } 5.$$

A. $\varphi_i = 0$ $(i = 1 \text{ to } 5)$.

Here our given equation is a quintic in x^2 with the roots x_i $(i = 1 \text{ to } 5)$. This quintic can be taken again in the reduced form and hence we may write throughout this section

$$\sum_{i=1}^{i=5} x_i^2 = 0.$$

The following Lagrange's system is the skeleton for our investigations in this section

$$\begin{aligned}
\psi_1 &= x_1 + \omega^3 x_2 + \omega x_3 + \omega^4 x_4 + \omega^2 x_5, \\
\psi_2 &= x_1 + \omega^4 x_2 + \omega^3 x_3 + \omega^2 x_4 + \omega x_5, \\
\psi_3 &= x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5, \\
\psi_4 &= x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega x_4 + \omega^3 x_5, \\
\psi_5 &= x_1 + x_2 + x_3 + x_4 + x_5.
\end{aligned}$$

Consider now for a moment the two systems of intransitivity by forming the function

$$y = \prod_{i=1}^{i=5} x_i,$$

which is changed by G_{10} into

$$-y = \prod_{i=1}^{i=5} x_{i+5}.$$

It follows then that y^2 is rationally known, and we have, if $y^2 = c$, the condition *c is rational, but not a perfect square* and $-c$ is of course the constant term in $f(x) = 0$ throughout part *A*.

Just as $\sqrt{c}$, so is

$$\psi_5 = \sum_{i=1}^{i=5} x_i$$

a function which is changed only by the odd substitutions of G_{10}; it follows then if

$$\frac{\psi_5}{\sqrt{c}} = 5r,$$

then *r is rationally known.*

Similarly ψ_i^5 ($i = 1$ to 4) takes only two values under G_{10}, so that

$$\frac{\psi_i^5}{\sqrt{c}}$$

is a quantity in our given rational domain if we adjoin to it ω, and we must have, for instance,

$$\psi_1^5 = \sqrt{c}[k + l\sqrt{5} + m\sqrt{-10+2\sqrt{5}} + n\sqrt{-10-2\sqrt{5}}],$$

where k, l, m, n are necessarily rational quantities.

Replacing finally k, l, m, n respectively by 5^5c^2K, $5^5\cdot c^2L$, 5^5c^2M, 5^5c^2N, then K, L, M, N are rationally known, and Lagrange's system becomes

$$\psi_i = 5\sqrt{c}\,\rho_i \qquad (i = 1 \text{ to } 4),$$

$$\rho_1 = \sqrt[5]{K + L\sqrt{5} + M\sqrt{-10+2\sqrt{5}} + N\sqrt{-10-2\sqrt{5}}},$$

$$\rho_2 = \sqrt[5]{K - L\sqrt{5} - N\sqrt{-10+2\sqrt{5}} + M\sqrt{-10-2\sqrt{5}}},$$

$$\rho_3 = \sqrt[5]{K - L\sqrt{5} + N\sqrt{-10+2\sqrt{5}} - M\sqrt{-10-2\sqrt{5}}},$$

$$\rho_4 = \sqrt[5]{K + L\sqrt{5} - M\sqrt{-10+2\sqrt{5}} - N\sqrt{-10-2\sqrt{5}}},$$

Combining the two forms of Lagrange's system, we get by addition the roots of $f(x) = 0$, namely

$$\begin{aligned}
x_1 &= -x_6 = +\sqrt{c}[\rho_1 + \rho_2 + \rho_3 + \rho_4 + r],\\
x_2 &= -x_7 = +\sqrt{c}[\omega^2\rho_1 + \omega\rho_2 + \omega^4\rho_3 + \omega^3\rho_4 + r],\\
x_3 &= -x_8 = +\sqrt{c}[\omega^4\rho_1 + \omega^2\rho_2 + \omega^3\rho_3 + \omega\rho_4 + r],\\
x_4 &= -x_9 = +\sqrt{c}[\omega\rho_1 + \omega^3\rho_2 + \omega^2\rho_3 + \omega^4\rho_4 + r],\\
x_5 &= -x_{10} = +\sqrt{c}[\omega^3\rho_1 + \omega^4\rho_2 + \omega\rho_3 + \omega^2\rho_4 + r].
\end{aligned}$$

It is seen now by direct computation that

$$\psi_1\psi_4 = 25c\sqrt[5]{P + Q\sqrt{5}} = (\omega + \omega^4)f_1 + (\omega^2 + \omega^3)f_2 = 25c\rho_1\rho_4,$$

and

$$\psi_2\psi_3 = 25c\sqrt[5]{P - Q\sqrt{5}} = (\omega^2 + \omega^3)f_1 + (\omega + \omega^4)f_2 = 25c\rho_2\rho_3,$$

where

$$\begin{aligned}
P &= K^2 + 5L^2 + 10(M^2 + N^2),\\
Q &= 2KL - 2(M^2 - N^2) - 8MN,\\
f_1 &= x_1x_3 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5,\\
f_2 &= x_1x_2 + x_1x_5 + x_2x_3 + x_3x_4 + x_4x_5.
\end{aligned}$$

Hence P and Q are rational and, as f_1 and f_2 are unchanged by our G_{10}, both f_1 and f_2 are rational too. It follows then that $\rho_1\rho_4$ and $\rho_2\rho_3$, $\psi_1\psi_4$ and $\psi_2\psi_3$ are rationally known if we adjoin $\sqrt{5}$ to our rational domain; for $(\omega^2 + \omega^3)$ as well as $(\omega + \omega^4)$ are quantities in $R(\sqrt{5})$. Therefore $P \pm Q\sqrt{5}$ is a perfect fifth power of a rational

quantity in this extended domain and hence we may write

$$\rho_1\rho_4 = R + S\sqrt{5} \quad \text{and} \quad \rho_2\rho_3 = R - S\sqrt{5},$$

where then R and S are rationally known and defined by

$$P = R^5 + 50R^3S^2 + 10RS^4,$$
$$Q = 5R^4S + 50R^2S^3 + 25S^5.$$

From our condition

$$\sum_1^5 x_i^2 = 0$$

follows by direct computation

$$r^2 + 2(\rho_1\rho_4 + \rho_2\rho_3) = 0,$$

hence

$$R = -\frac{r^2}{4}.$$

Whenever our given rational domain would not include any imaginary quantity, then $R = r = 0$, for if $R < 0$ then under our condition $P < 0$ but P is the sum of squares.

We take up this natural division here and write

A. I. $r = 0$. A. II. $r \neq 0$.

A. I. $r = R = P = \psi_5 = 0 \;\therefore\; \rho_1\rho_4 = -\rho_2\rho_3$

This case will be subdivided again, namely into

1. $S = 0$. 2. $S \neq 0$.

1. If $S = 0$, then $\rho_1\rho_4 = \rho_2\rho_3 = 0$ and we may take without losing any generality

$$\rho_1 = \rho_2 = 0.$$

From $\rho_1 = 0$ follows:

$$K + L\sqrt{5} + M\sqrt{-10 + 2\sqrt{5}} + N\sqrt{-10 - 2\sqrt{5}} = 0,$$

and, from $\rho_2 = 0$

$$K - L\sqrt{5} - N\sqrt{-10 + 2\sqrt{5}} + M\sqrt{-10 - 2\sqrt{5}} = 0,$$

hence

$$\psi_3 = 5\sqrt{c}\sqrt[5]{2(K - L\sqrt{5})} \quad \text{and} \quad \psi_4 = 5\sqrt{c}\sqrt[5]{2(K + L\sqrt{5})}.$$

By addition we get further

$$2K + (M - N)\sqrt{-10 + 2\sqrt{5}} + (M + N)\sqrt{-10 - 2\sqrt{5}} = 0,$$

or

$$2K + \sqrt{-10 + 2\sqrt{5}}\left[M - N + (M + N)\frac{4\sqrt{5}}{-10 + 2\sqrt{5}}\right] = 0.$$

The condition

$$\prod_{i=1}^{i=5} x_i = \sqrt{c}$$

shows

$$K = \left(\frac{1}{2c}\right)^2,$$

and hence $K \neq 0$, so that the coefficient of $\sqrt{-10+2\sqrt{5}}$ cannot vanish and ω *belongs to our rational domain* for with $\sqrt{-10+2\sqrt{5}}$ is $\sqrt{5}$ rationally known and hence also

$$\sqrt{-10-2\sqrt{5}} = \frac{4\sqrt{5}}{\sqrt{-10+2\sqrt{5}}}$$

As $K \neq 0$, both ψ_3 and ψ_4 cannot vanish; in fact, if

$$\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$$

then

$$x_1 = x_2 = x_3 = x_4 = x_5$$

and $f(x)$ is reducible. We may, however, very well have that three ρ's vanish and we again take a subdivision without losing any generality by writing

$$\alpha.\ \rho_3 = 0; \quad \rho_4 \neq 0. \qquad \beta.\ \rho_3 \neq 0; \quad \rho_4 \neq 0.$$

α. If $\rho_3 = 0$, then

$$K = L\sqrt{5} = \frac{1}{4c^2}$$

and

$$\rho_4 = \sqrt[5]{1/c^2}.$$

In this case the ten roots of $f(x) = 0$ are

$$\begin{aligned}
x_1 &= -x_6 = \sqrt{c}\,\rho_4 = \sqrt[10]{c},\\
x_2 &= -x_7 = \omega^2\sqrt[10]{c},\\
x_3 &= -x_8 = \omega^4\sqrt[10]{c},\\
x_4 &= -x_9 = \omega\sqrt[10]{c},\\
x_5 &= -x_{10} = \omega^3\sqrt[10]{c}.
\end{aligned}$$

The corresponding equation is

$$f(x) = x^{10} - c = 0,$$

and hence *c is not a perfect fifth power in our domain*, for then $f(x)$ splits up into five rational factors each of the second degree.

Our next question is: Are the derived conditions also sufficient that $f(x) = 0$ represents an irreducible Abelian equation? In this simple case it is readily seen from the above roots that $x^{10} - c = 0$ is irreducible under the derived conditions. Adjoining $\sqrt{c}$ to the given rational domain, then $x^{10} - c$ splits up into two rational factors $(x^5 - \sqrt{c})$ and $(x^5 + \sqrt{c})$; adjoining $\sqrt[5]{c}$ to our given domain, then $x^{10} - c$ splits up into five rational factors, each of the type $(x^2 - \omega^i\sqrt[5]{c})$. We see therefore that the Galois group of $f(x) = 0$ can only be a group of type 3 in Cole's list.* In fact the group is G_{10}, for the ten roots in our cycle

$$s = (1739562840)$$

satisfy the condition, that every root is the same rational function of the preceding one in the cycle, namely

$$x_1 = -\omega^3 x_{10}; \quad x_7 = -\omega^3 x_1; \quad x_3 = -\omega^3 x_7, \quad \text{etc.},$$

ω being rational by condition.

* *Loc. cit.*

Type I: If c is neither a perfect fifth nor a perfect second power of any quantity in the given rational domain and if ω, c, are in this domain, then $x^{10} - c = 0$ is an irreducible Abelian equation having the above roots.

An example is furnished by taking $c = 2$. Here $\sqrt{-10 + 2\sqrt{5}}$ must be adjoined to $R(1)$.

$$f(x) \equiv x^{10} - 2 \equiv (x^5 - \sqrt{2})(x^5 + \sqrt{2})$$
$$= (x^2 - \sqrt[5]{2})(x^2 - \omega\sqrt[5]{2})(x^2 - \omega^2\sqrt[5]{2})(x^2 - \omega^3\sqrt[5]{2})(x^2 - \omega^4\sqrt[5]{2}) = 0$$

is an irreducible Abelian equation under the above conditions.

$$\beta. \quad \rho_3 \neq 0; \quad \rho_4 \neq 0.$$

The function

$$\frac{\psi_3^2}{\psi_4} = \frac{(x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5)^2}{x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega x_4 + \omega^3 x_5}$$

is a two-valued function under G_{10} just as $\sqrt{c}$; hence, writing

$$\frac{\psi_3^2}{\psi_4\sqrt{c}} = 5e,$$

then *e is rationally known but not zero,* and we have

$$25c\sqrt[5]{4(K - L\sqrt{5})^2} = 25e\cdot c\sqrt[5]{2(K + L\sqrt{5})},$$

or, as

$$K = \frac{1}{4c^2},$$

$$2^4\cdot 5L^2c^4 - 2^3L\sqrt{5}c^2(1 + e^5c^2) + (1 - 2e^5c^2) = 0,$$

which determines L for $L \neq 0$, as then

$$e = \sqrt[5]{2K} = \sqrt[5]{1/2c^2}$$

and

$$(x - x_1)(x - x_6) = x^2 - 4ce^2$$

would be a rational factor of $f(x) = 0$.

Solving for L we get

$$L = \frac{\sqrt{5}}{20c^2}[1 + e^5c^2 \pm e^2c\sqrt{e^6c^2 + 4e}];$$

as L is rationally known it follows $e^6c^2 + 4e$ *is a perfect square of a rational quantity.* Observing that

$$\rho_3^2 = \sqrt[5]{2^2(+K - L\sqrt{5})^2},$$

we have to choose the negative sign in the expression for L and hence

$$L = \frac{\sqrt{5}}{20c^2}[1 + e^5c^2 - e^2c\sqrt{e^6c^2 + 4e}].$$

So we have

$$\rho_3 = \sqrt[5]{-\frac{e^5}{2} + \frac{e^2}{2c}\sqrt{e^6c^2 + 4e}},$$

$$\rho_4 = \sqrt[5]{\frac{1}{c^2} + \frac{e^5}{2} - \frac{e^2}{2c}\sqrt{e^6c^2 + 4e}},$$

and $f(x) = 0$ becomes

$$f(x) \equiv x^{10} - 10c^2\rho_3{}^3\rho_4x^6 - 25\rho_3{}^2\rho_4{}^4c^3x^4 + (15\rho_3{}^6\rho_4{}^2 - 10\rho_3\rho_4{}^7)c^4x^2 - c = 0.$$

All coefficients of $f(x)$ are rational at the same time when

$$e = \frac{\rho_3{}^2}{\rho_4}$$

is rational, for as ω is in our domain $\rho_3{}^5$ and $\rho_4{}^5$ are rationally known; hence $\frac{\rho_3{}^2}{\rho_4} \cdot \rho_4{}^5$ and therefore $\rho_3{}^3\rho_4$ is rationally known; but then all powers of $\rho_3{}^3\rho_4$ are rational, so that $\rho_3\rho_4{}^2$; $\rho_3{}^4\rho_4{}^3$; $\rho_3{}^2\rho_4{}^4$ are in our given domain. Further $\rho_3 \cdot \rho_4$ is irrational, for if it were rational

$$\frac{\rho_3\rho_4{}^2}{\rho_3\rho_4} = \rho_4,$$

and hence ρ_3 are rationally known, but then $(x - x_1)(x - x_6)$ would be a rational factor of $f(x)$. It follows then that

$$-\frac{e^5}{2} + \frac{e^2}{2c}\sqrt{e^6c^2 + 4e}$$

is not a perfect fifth power of a rational quantity.

The roots of $f(x) = 0$ are now

$$\begin{aligned}
x_1 &= -x_6 = \sqrt{c}(\rho_3 + \rho_4),\\
x_2 &= -x_7 = \sqrt{c}(\omega^4\rho_3 + \omega^3\rho_4),\\
x_3 &= -x_8 = \sqrt{c}(\omega^3\rho_3 + \omega\rho_4),\\
x_4 &= -x_9 = \sqrt{c}(\omega^2\rho_3 + \omega^4\rho_4),\\
x_5 &= -x_{10} = \sqrt{c}(\omega\rho_3 + \omega^2\rho_4),
\end{aligned}$$

where

$$\rho_3 = \sqrt[5]{-\frac{e^5}{2} + \frac{e^2}{2c}\sqrt{e^6c^2 + 4e}},$$

$$\rho_4 = \sqrt[5]{\frac{1}{c^2} + \frac{e^5}{2} - \frac{e^2}{2c}\sqrt{e^6c^2 + 4e}}.$$

Regarding the sufficiency of the derived conditions, we note first that the sum of any two roots is either zero or irrational as ω is rational and both ρ_3 and ρ_4 are irrational. The sum of two roots can be zero only if their subscripts differ by 5, in which case however the product $x_i \cdot x_{i+5}$ $(i = 1 \text{ to } 5)$ is irrational, for it is of the form

$$c(\omega^i\rho_3 + \omega^j\rho_4)^2,$$

and must contain the irrationalities ρ_3 and ρ_4 by conditions. It follows then, that $f(x)$ cannot contain a rational factor of the second degree. Similarly it is seen that by means of our conditions there is no rational factor of the fourth degree, for the product of any four roots is a polynomial of the fourth degree in the ρ's with rational coefficients, and this cannot be rationally known, as the ρ's satisfy an irreducible quintic. There is certainly no rational factor of odd degree, for the constant term would contain $\sqrt{c}$ as a factor, which cannot be expressed rationally in terms of the ρ's. It follows then that $f(x)$ is irreducible and its Galois

group is transitive. Furthermore $f(x)$ contains two as well as five systems of intransitivity, for the factors $(x - x_i)(x - x_{i+5})$ are free from $\sqrt{c}$ and the two factors

$$\prod_{i=1}^{i=5}(x + x_i) \quad \text{and} \quad \prod_{i=1}^{i=5}(x - x_{i+5})$$

are free from ρ_3 and ρ_4 and contain only those combinations of ρ_3 and ρ_4 or those powers of ρ_3 and ρ_4 which have been seen to be rationally known; for we have

$$\sum_{i=1}^{i=5} x_i = 0,$$

$$\sum_{\substack{i=1\\j=2}}^{\substack{i=4\\j=5}} x_i x_j = 0 \quad (i < j),$$

$$_{i,j,k}\sum_{\substack{1\\i<j<k}}^{5} x_i x_j x_k = 5c\sqrt{c}\rho_4^2\rho_3,$$

$$_{i,j,k,l}\sum_{\substack{1\\i<j<k<l}}^{5} x_i x_j x_k x_l = -5c^2\rho_3^3\rho_4,$$

$$\prod_{i=1}^{i=5} x_i = \sqrt{c}.$$

The Galois group of our above $f(x) = 0$ is therefore a group of type 3. Next we show that neither G_{10}' nor G_{20}' can be a subgroup of this Galois group. Applying

$$\sigma_1 = (16)(20)(39)(48)(57)$$

as well as

$$\sigma_2 = (1748)(2936)(50)$$

to the function

$$5e = \frac{(x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5)^2}{\sqrt{c}(x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega x_4 + \omega^3 x_5)} \neq 0$$

we change $5e$ to zero, which is contrary to condition. σ_1 as well as σ_2 cannot belong to the group of $f(x) = 0$ under our derived conditions and hence the group of $f(x) = 0$ is G_{10}.

Type II. If c, e, ω, $\sqrt{e^6c^2 + 4e}$ but not

$$\sqrt[5]{-\frac{e^5}{2} + \frac{e^2}{2c}\sqrt{e^6c^2 + 4e}} \quad \textit{and} \quad \sqrt{c}$$

are quantities in our given domain and if $e \neq 0$, then the above equation $f(x) = 0$ with the roots on page 124 is an irreducible Abelian equation.

An example is furnished by taking $e = 1$, $c = \frac{3}{2}$; then

$$4e + c^2e^6 = \left(\frac{5}{2}\right)^2 \qquad \rho_3 = \sqrt[5]{1/3} \qquad \rho_4 = \sqrt[5]{1/9}$$

$$f(x) = x^{10} - \frac{15}{2}x^6 - \frac{75}{8}x^4 + \frac{105}{16} - x^2\,\frac{3}{2} = 0.$$

The roots are

$$x_1 = -x_6 = \sqrt{3/2}[\sqrt[5]{1/3} + \sqrt[5]{1/9}],$$
$$x_2 = -x_7 = \sqrt{3/2}[\omega^4 \sqrt[5]{1/3} + \omega^3 \sqrt[5]{1/9}],$$
$$\cdot \quad \cdot \quad \cdot \quad \text{etc.} \quad \cdot \quad \cdot \quad \cdot$$

As given domain may be taken $R(\omega)$.

2. If $S \neq 0$ and hence $\rho_i \neq 0$ ($i = 1$ to 4), we shall find that there is no Abelian type.

Actual computation shows

$$\psi_1^2\psi_2 + \psi_4^2\psi_3 = \sum_{i=1}^{i=5} x_i^3 - \frac{3+\sqrt{5}}{2}F_1 - \frac{3-\sqrt{5}}{2}F_2 + 2(1+\sqrt{5})F_3 + 2(1-\sqrt{5})F_4$$

and

$$\psi_3^2\psi_1 + \psi_2^2\psi_4 = \sum_{i=1}^{i=5} x_i^3 - \frac{3-\sqrt{5}}{2}F_1 - \frac{3+\sqrt{5}}{2}F_2 + 2(1-\sqrt{5})F_3 + 2(1+\sqrt{5})F_4,$$

where

$$F_1 = x_1^2x_2 + x_2^2x_1 + x_1^2x_5 + x_5^2x_1 + x_2^2x_3 + x_3^2x_2 + x_4^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_4,$$
$$F_2 = x_1^2x_3 + x_3^2x_1 + x_1^2x_4 + x_4^2x_1 + x_2^2x_4 + x_4^2x_2 + x_2^2x_5 + x_5^2x_2 + x_3^2x_5 + x_5^2x_3,$$
$$F_3 = x_1x_2x_3 + x_1x_2x_5 + x_1x_4x_5 + x_2x_3x_4 + x_3x_4x_5,$$
$$F_4 = x_1x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_5 + x_2x_4x_5.$$

All these four functions are double-valued under G_{10} just as $\sqrt{c}$. It follows, then, if λ_1 and λ_2 are in our given rational domain, that we may write

$$\rho_1^2\rho_2 + \rho_4^2\rho_3 = \lambda_1 + \lambda_2\sqrt{5}, \tag{1}$$
$$\rho_2^2\rho_4 + \rho_3^2\rho_1 = \lambda_1 - \lambda_2\sqrt{5}. \tag{2}$$

λ_1 and λ_2 cannot both vanish at once, for then, observing that

$$\rho_1\rho_4 = +S\sqrt{5}, \tag{3}$$
$$\rho_2\rho_3 = -S\sqrt{5}, \tag{4}$$

we would have

$$-\rho_4^4 = \rho_2^3\rho_3 \quad \text{and} \quad \frac{1}{\rho_4^2} = \frac{\rho_2}{\rho_3^3};$$

hence, by proper multiplication of the last two,

$$-1 = \frac{\rho_2^5}{\rho_3^5} \quad \text{or} \quad \rho_2 = -\rho_3,$$

and hence (2) appears in the form $\rho_1 + \rho_4 = 0$ so that $f(x)$ would contain the rational factor

$$(x - x_1)(x - x_6) = x^2.$$

Using equations (3) and (4) in order to eliminate ρ_2 and ρ_4 in (1) and (2) and eliminating finally ρ_3, we get

$$\frac{\rho_1^5G^2}{100S^4} = \tfrac{1}{2}[(\lambda_1 - \lambda_2\sqrt{5}) \pm \sqrt{(\lambda_1 - \lambda_2\sqrt{5})^2 - 20S^3\sqrt{5}}], \tag{5}$$

where

$$G = \lambda_1 + \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 + \lambda_2\sqrt{5})^2 + 20S^3\sqrt{5}}.$$

Eliminating ρ_3 and ρ_4 in (1) and (2) by means of (3) and (4) and finally between these two new equations (1a) and (1b) eliminating ρ_2 we get

$$\frac{4\rho_1^5}{G^2} = \frac{\lambda_1 - \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 - \lambda_2\sqrt{5})^2 - 20S^3\sqrt{5}}}{10S^2}. \tag{6}$$

Comparing (5) and (6) we have

$$G^4 = 2000S^6, \tag{7}$$

and hence there are two possibilities. If

$$(\lambda_1 + \lambda_2\sqrt{5})[\lambda_1 + \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 + \lambda_2\sqrt{5})^2 + 20S^3\sqrt{5}}] = 0,$$

then

$$\lambda_1 + \lambda_2\sqrt{5} = 0, \tag{8}$$

for if the other factor were zero we would have $S = 0$. If

$$(\lambda_1 + \lambda_2\sqrt{5})[\lambda_1 + \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 + \lambda_2\sqrt{5})^2 + 20S^3\sqrt{5}}] = -20S^3\sqrt{5},$$

then

$$(\lambda_1 + \lambda_2\sqrt{5})^2 = -20S^3\sqrt{5}. \tag{9}$$

Doing with ρ_2 exactly as has just been done with ρ_1 we are led to the following result: We must have
either

$$\lambda_1 - \lambda_2\sqrt{5} = 0 \tag{10}$$

or

$$(\lambda_1 - \lambda_2\sqrt{5})^2 = +20S^3\sqrt{5}. \tag{11}$$

One of the two relations (8) and (9) must exist with one of the two relations (10) and (11). As we saw, $\lambda_1 = \lambda_2 = 0$ is impossible, hence we cannot pair (8) and (10). Also relation (9) cannot exist with relation (11), for we would get

$$\lambda_1^2 + 5\lambda_2^2 = 0,$$

which leads to $S = 0$. It follows that either (8) and (11) or (9) and (10) exist together. Both cases are identical in the abstract; they differ only in the sign of $\sqrt{5}$. Assuming then

$$\lambda_1 + \lambda_2\sqrt{5} = 0, \tag{8}$$

we have from (11)

$$\lambda_1^2 = 5S^3\sqrt{5}, \tag{11a}$$

and $\sqrt{5}$ is rational and further $\sqrt{S\sqrt{5}}$ is in our domain. Equations (1) and (2) appear now in the form

$$\rho_1^2\rho_2 = -\rho_4^2\rho_3, \tag{12}$$

and

$$\rho_2^2\rho_4 + \rho_3^2\rho_1 = 2\lambda_1, \tag{13}$$

which may be written, by observing (3) and (4),

$$\rho_2^2\rho_4 + \frac{5S^3\sqrt{5}}{\rho_2^2\rho_4} = 2\lambda_1,$$

or, by means of (11*a*),

$$\lambda_1 = \rho_2^2\rho_4;$$

hence from (13) we have

$$\lambda_1 = \rho_2^2\rho_4 = \rho_3^2\rho_1;$$

hence

$$\rho_1 = \frac{\rho_2^2\rho_4}{\rho_3^2}. \tag{14}$$

From (12) we get, by means of (3), (4), and (11*a*)

$$\rho_1^2\rho_2 = + \frac{\lambda_1^2}{\rho_1^2\rho_2};$$

hence

$$\rho_1^2\rho_2 = \pm \lambda_1,$$

and we have finally

$$\lambda_1 = \rho_2^2\rho_4 = \rho_3^2\rho_1 = \pm \rho_1^2\rho_2 = \mp \rho_4^2\rho_3, \tag{15}$$

so that

$$\rho_1 = \pm \frac{\rho_3^2}{\rho_2} \tag{16}$$

and

$$\rho_1 = \mp \frac{\rho_4^2}{\rho_3}. \tag{17}$$

Observing (14), (16), (17) we may write

$$\pm \rho_1^2 = \frac{\rho_2^2\rho_4}{\rho_3^2} \cdot \frac{\rho_3^2}{\rho_2} = \rho_2\rho_4,$$

$$\pm \rho_1^3 = \left(\frac{\rho_4^2}{\rho_3}\right)^2 \cdot \frac{\rho_3^2}{\rho_2} = \frac{\rho_4^4}{\rho_2}.$$

Multiplying the last two relations together we obtain

$$\rho_1^5 = \rho_4^5 \quad \text{or} \quad \rho_1 = \rho_4, \tag{18}$$

so that

$$\sqrt{\rho_1\rho_4} = \rho_1 = \rho_4 = \sqrt{S\sqrt{5}}$$

is rationally known, as we saw above. With ρ_1 and ρ_4 are ρ_2 and ρ_3 rationally known by means of (15) and hence

$$(x - x_1)(x - x_6)$$

is a rational factor of $f(x)$. There is no irreducible Abelian type under A. I. 2.

A. II. $r \neq 0$.

We had the relation

$$r^2 + 2(\rho_1\rho_4 + \rho_2\rho_3) = 0,$$

so that at most two ρ's can vanish, either $\rho_1 = \rho_4 = 0$ or $\rho_2 = \rho_3 = 0$. Without

loss of generality we may take the following division of case II, $\rho_1 \neq 0$; $\rho_4 \neq 0$ and

$$(1)\ \rho_2 = \rho_3 = 0, \qquad (2)\ \rho_2 = 0;\quad \rho_3 \neq 0, \qquad (3)\ \rho_2 \neq 0;\quad \rho_3 \neq 0.$$

1. If $\rho_2 = \rho_3 = 0$ then

$$K = L\sqrt{5}$$

and

$$N = -M\frac{1+\sqrt{5}}{2}.$$

Suppose for a moment $K = L = 0$ then $\rho_1 = -\rho_4$ so that $f(x)$ contains the rational factor $(x^2 - cr)$. It follows then

$$K = L\sqrt{5} \neq 0$$

and hence $\sqrt{5}$ *is in our given domain of rationality.*

We may now write

$$\rho_1{}^5 = 2K + M'\sqrt{-10 + 2\sqrt{5}},$$
$$\rho_4{}^5 = 2K - M'\sqrt{-10 + 2\sqrt{5}},$$

where

$$M' = \frac{5+\sqrt{5}}{2}M$$

is necessarily rationally known.

$$(\rho_1\rho_4)^5 = -\left(\frac{r^2}{2}\right)^5 = 4K^2 - M'^2(-10 + 2\sqrt{5}),$$

so that M' can be determined by c and r as K is found from the condition

$$\prod_{i=1}^{i=5} x_i = \sqrt{c},$$

which leads to the relation

$$K = \frac{1 - \frac{19}{4}r^5c^2}{4c^2};$$

hence $r^5c^2 \neq \frac{4}{19}$ as $K \neq 0$ and $c \neq 0$.

$$M' = \frac{1}{8c^2}\sqrt{\frac{16 - 152r^5c^2 + 363r^{10}c^4}{-10 + 2\sqrt{5}}} \neq 0,$$

for if $M' = 0$, then $M = N = 0$ and $\rho_1 = -\rho_4$, so that $\rho_1\rho_4 = -\rho_1{}^2 = -\rho_4{}^2$ and therefore ρ_1, ρ_4 would be rational and hence $f(x)$ is reducible. It follows then

$$\sqrt{\frac{16 - 152r^5c^2 + 363r^{10}c^4}{-10 + 2\sqrt{5}}} \neq 0,$$

but is in our rational domain.

The roots of $f(x) = 0$ are given on page 120, where the ρ's are defined by

$$\rho_1 = \sqrt[5]{\frac{1 - \frac{19}{4} r^5 c^2}{2c^2} + \frac{1}{8c^2}\sqrt{16 - 152 r^5 c^2 + 363 r^{10} c^4}},$$

$$\rho_2 = \rho_3 = 0,$$

$$\rho_4 = \sqrt[5]{\frac{1 - \frac{19}{4} r^5 c^2}{2c^2} - \frac{1}{8c^2}\sqrt{16 - 152 r^5 c^2 + 363 r^{10} c^4}}.$$

The corresponding equation becomes

$$f(x) = x^{10} + \frac{115}{4} r^4 c^2 x^6 + 5\left(\frac{15}{2} r^6 - 2\frac{r}{c^2}\right) c^3 x^4 + \left(\frac{541}{16} r^8 - \frac{35}{c^2} r^3\right) c^4 x^2 - c = 0.$$

We ask again: Are the derived conditions also sufficient, that $f(x) = 0$ is an irreducible Abelian equation?

Our $f(x)$ cannot have any rational factor of odd degree, for the constant term would contain $\sqrt{c}$ as a factor which is irrational by conditions. $\sqrt{c}$ and ρ_i cannot be expressed rationally by each other. Further, as every root is of the form

$$\pm \sqrt{c}[\omega^k \rho_1 + \omega^{-k}\rho_4 + r] = \pm \sqrt{c}\left[\omega^k \rho_1 - \frac{r^2}{2\omega^k \rho_1} + r\right],$$

the sum of any two roots assumes one of the two forms either

$$(1) \qquad \pm \sqrt{c}\left[\rho_1(\omega^k + \omega^j) - \frac{r^2(\omega^k + \omega^j)}{2\rho_1 \omega^{k+j}} + 2r\right],$$

or

$$(2) \qquad \pm \sqrt{c}\left[\rho_1 - \frac{r^2}{2\rho_1 \omega^{k-j}}\right](\omega^k - \omega^j).$$

The type (2) shows that, if $k = j$, then the sum of the two roots vanishes, which can happen only if the subscripts differ by five. In this case, however, the product $x_i x_{i+5}$ contains the irrationality ρ_1, for we have

$$x_i x_{i+5} = -c\left[\omega^k \rho_1 - \frac{r^2}{2\omega^k \rho_1} + r\right]^2 \neq 0.$$

In all other cases the sum of any two roots does not vanish and hence both irrationalities remain in the expression of the sum, for if in (2)

$$\rho_1 = \frac{r^2}{2\rho_1 \omega^{k-j}},$$

then

$$r = \rho_1 \omega^{\mu} \sqrt{2},$$

which cannot happen, as ρ_1 satisfies an irreducible quintic, $\sqrt{2}$ a quadratic and ω an irreducible quadratic for $\sqrt{5}$ is rational. If in (1)

$$\rho_1(\omega^k + \omega^j) - \frac{r^2(\omega^k + \omega^j)}{2\rho_1 \omega^{k+j}} = 0,$$

then

$$\rho_1^2 = \left(\frac{r}{\omega^\lambda}\right)^2 \cdot \frac{1}{2},$$

which would be again impossible, as ρ_1 is irrational and satisfies an irreducible quintic.

Exactly the same thing can be shown in the case of a factor of the fourth degree, having now instead of $\rho_1(\omega^k + \omega^j)$ and ω^{k+j}, always $\rho_1(\omega^k + \omega^j + \omega^l + \omega^m)$ and $\omega^{k+j+l+m}$ respectively. If

$$\omega^k + \omega^l + \omega^j + \omega^m = -1,$$

then

$$\omega^{k+l+j+m} = +1,$$

and the sum of the four roots vanishes; but then the product of the four roots is irrational for the same reason as in the case $x_i x_{i+5}$.

Whenever, then, ρ_1 is irrational, we see, that the above conditions are sufficient for the irreducibility of $f(x)$. Hence we write

$$\sqrt[5]{\frac{1 - \frac{19}{4} r^5 c^2}{2c^2} + \frac{1}{8c^2}\sqrt{\frac{16 - 152c^2r^5 + 363c^4r^{10}}{-10 + 2\sqrt{5}}}}$$

is irrational.

Under the above conditions, then, the Galois group is transitive and must be again of type 3 in Cole's list for the five factors

$$(x - x_i)(x - x_{i+5}) \quad (i = 1 \text{ to } 5)$$

are rational after the adjunction of ρ_1 to the rational domain and the two factors

$$\prod_{i=1}^{i=5} (x - x_i) \quad \text{and} \quad \prod_{i=1}^{i=5} (x - x_{i+5})$$

are rational after the adjunction of $\sqrt{c}$ to the given domain of rationality for we find:

$$\sum_{i=1}^{=5} x_i = 0,$$

$$\sum_{\substack{1 \\ i<j}}^{5} {}_{i,j}\, x_i x_j = \frac{25}{2} r^2 c,$$

$$\sum_{\substack{1 \\ i<j<k}}^{5} {}_{i,j,k}\, x_i x_j x_k = \frac{35}{2} r^3 c \sqrt{c},$$

$$\sum_{\substack{1 \\ i<j<k<l}}^{5} {}_{i,j,k,l}\, x_i x_j x_k x_l = \frac{45}{4} r^4 c^2,$$

$$\sum_{i=1}^{i=5} x_i = +\sqrt{c}.$$

Now σ_1 interchanges $\pm \psi_1$ and $\mp \psi_4$, from which would follow that $\rho_1 = \pm \rho_4$, if σ_1 were in our Galois group, and hence $K = 0$ or $M' = 0$, which is contrary to our conditions.

σ_2 changes $\psi_1\psi_4 \neq 0$ to zero which is impossible for any substitution in our

Galois group, as $\psi_1\psi_4$ is rationally known. It follows then that our $f(x) = 0$ is an irreducible Abelian equation.

Type III. If c, $\sqrt{5}$, r but not $\sqrt{c}$ are in our given rational domain, and if

$$\sqrt{\frac{16 - 152r^5c^2 + 363r^{10}c^4}{-10 + 2\sqrt{5}}} \neq 0$$

but rational and

$$\sqrt[5]{\frac{1 - \frac{19}{4}r^5c^2}{2c^2} + \frac{1}{8c^2}\sqrt{\frac{16 - 152r^5c^2 + 363c^4r^{10}}{-10 + 2\sqrt{5}}}}$$

is irrational and $r \neq 0$, $r^5c^2 \neq \frac{4}{19}$ then the above $f(x)$ is an irreducible Abelian equation with roots as given above.

An example is furnished by taking $r = 1$; $c = 2$. Then

$$\frac{16 - 152c^2r^5 + 363c^4r^{10}}{-10 + 2\sqrt{5}} = \frac{16 \cdot 163}{-5 + \sqrt{5}};$$

hence our rational domain must contain

$$\sqrt{\frac{163}{-5 + \sqrt{5}}}$$

$$f(x) = x^{10} + 115x^6 + 280x^4 + 401x^2 - 2 = 0$$

The roots are

$$x_1 = -x_6 = +\sqrt{2}\left[\sqrt[5]{-\frac{9}{4} + \frac{1}{8}\sqrt{326}} + \sqrt[5]{-\frac{9}{4} - \frac{1}{8}\sqrt{326}} + 1\right],$$

$$x_2 = -x_7 = +\sqrt{2}\left[\omega^2\sqrt[5]{-\frac{9}{4} + \frac{1}{8}\sqrt{326}} + \omega^3\sqrt[5]{-\frac{9}{4} - \frac{1}{8}\sqrt{326}} + 1\right],$$

. . . .

etc.

2. If $\rho_2 = 0$, $\rho_i \neq 0$ $(i = 1, 3, 4)$, we have

$$\rho_3 = \sqrt[5]{2(K - L(\sqrt{5})} \neq 0,$$

which follows from $\rho_2 = 0$, that is, from

$$K - L\sqrt{5} = N\sqrt{-10 + 2\sqrt{5}} - M\sqrt{-10 - 2\sqrt{5}},$$

so that ω *is rationally known.*

The functions

$$\frac{\psi_3^2}{\psi_4\sqrt{c}} \quad \text{and} \quad \frac{\psi_1^2}{\psi_3\sqrt{c}}$$

are seen to be rationally known, if applied to by G_{10}. Calling them $5e$ and $5d$

respectively we have

$$e = \frac{\rho_3^2}{\rho_4} \quad \text{and} \quad d = \frac{\rho_1^2}{\rho_3},$$
$$\rho_1^5 = -\frac{ed^2r^2}{2},$$
$$\rho_3^5 = +\frac{e^2r^4}{4d},$$
$$\rho_4^5 = +\frac{r^8}{2^4ed^2}.$$

If ρ_1 were rational, then ρ_4 and ρ_3 are rational and vice versa; hence we have the condition e, d, *are rationally known and* $2^4ed^2r^2$ *is not a perfect fifth power.* The last condition excludes $e = 0$, and $d = 0$ as well as $r = 0$. The corresponding equation becomes

$$f(x) = x^{10} + 5A_1r^2c^2x^6 + 5A_2r^3c^3x^4 + \frac{5}{2}A_3r^4c^4x^2 - c = 0,$$

where

$$A_1 = \frac{23}{4}r^2 + de + 3re - \frac{1}{2}\frac{r^2e}{d} - \frac{3}{2}\frac{r^3}{d},$$
$$A_2 = -2r^3 + d^2e - \frac{e^2r^2}{d} - \frac{r^6}{8d^2e} - 5r^2e + \frac{15}{2}rde - \frac{5}{2}\frac{r^3e}{d} - \frac{5}{4}re^2 - \frac{5}{16}\frac{r^5}{d^2},$$
$$A_3 = \frac{71}{8}r^4 + \frac{1}{2}\frac{r^2e^3}{d} - \frac{r^8}{16d^3e} + \frac{3d^2e^2}{2} + \frac{3r^4e^2}{8d^2} + \frac{3r^6}{4d^2} - 7r^2de - \frac{5}{4}\frac{r^4e}{d} - \frac{7}{8}\frac{r^7}{d^2e},$$
$$+ 7rd^2e - \frac{r^7e^2}{d} - \frac{7}{2}\frac{r^5}{d} - \frac{51}{2}r^3e + 5r^2e^2 - 5re^2d - \frac{5}{4}\frac{r^5e}{d^2}.$$

The roots are—

$$x_1 = -x_6 = +\sqrt{c}\left[-\sqrt[5]{\frac{ed^2r^2}{2}} + \sqrt[5]{\frac{e^2r^4}{4d}} + \sqrt[5]{\frac{r^8}{16ed^2}} + r\right],$$
$$x_2 = -x_7 = +\sqrt{c}\left[-\omega^2\sqrt[5]{\frac{ed^2r^2}{2}} + \omega^4\sqrt[5]{\frac{e^2r^4}{4d}} + \omega^3\sqrt[5]{\frac{r^8}{16ed^2}} + r\right],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

etc.

r, c, e, d are not independent of each other, for they have to satisfy the identity

$$\frac{1}{c^2} = \frac{19}{4}r^5 + \frac{r^8}{16ed^2} + \frac{r^4e^2}{4d} - \frac{ed^2r^2}{2} + \frac{15}{8}\frac{r^6}{d} + \frac{15}{4}r^4e + \frac{5}{2}r^3de - \frac{5}{4}\frac{r^5e}{d},$$

which follows from the condition

$$\prod_{i=1}^{i=5} x_i = \sqrt{c}.$$

Regarding the sufficiency of the derived conditions, we first note that $f(x)$ cannot have a rational factor of odd degree, for the constant term would contain $\sqrt{c}$ as a factor and $\sqrt{c}$ cannot be expressed rationally by ρ_i $(i = 1, 3, 4)$. Whenever the sum of any two or four roots vanishes, then these sums must be of the form

$$x_i + x_{i+5} \quad \text{or} \quad x_i + x_{i+5} + x_j + x_{j+5} \quad (i \neq j),$$

in which cases the product of the two or four roots is of the form

$$-c[\omega^k\rho_1 + \omega^l\rho_3 + \omega^m\rho_4 + r]^2 \qquad \text{or}$$
$$+ c^2[\omega^r\rho_1 + \omega^s\rho_3 + \omega^t\rho_4 + r]^2 \cdot [\omega^k\rho_1 + \omega^l\rho_3 + \omega^m\rho_4 + r]^2.$$

In each bracket stands an expression satisfying an irreducible quintic and hence only a power of $s \equiv 0 \pmod{5}$ can make the product rational. Both products are of lower degree than five in the ρ's and hence they are irrational. Whenever the sum of any two or four roots does not vanish, it must contain $\sqrt{c}$ as a factor. It follows, then, that $f(x)$ is irreducible; therefore its Galois group is transitive. This group must contain again two as well as five systems of intransitivity, for the five products

$$(x - x_i)(x - x_{i+5}) \quad (i = 1 \text{ to } 5).$$

do not show the irrationality $\sqrt{c}$, and the two products

$$\prod_{i=1}^{i=5}(x - x_i) \quad \text{and} \quad \prod_{i=1}^{i=5}(x - x_{i+5})$$

do not show the irrationality ρ, for we have

$$\sum_{i=1}^{i=5} x_i = 5r\sqrt{c},$$

$$\sum_{\substack{i,j=1\\ i<j}}^{i,j=5} x_ix_j = +\frac{25}{2}r^2c,$$

$$\sum_{\substack{i,j,k=1\\ i<j<k}}^{i,j,k=5} x_ix_jx_k = \frac{5cr^2\sqrt{c}}{4}\left[14r - 2e + \frac{r^2}{d}\right],$$

$$\sum_{\substack{i,j,k,l=1\\ i<j<k<l}}^{i,j,k,l=5} x_ix_jx_kx_l = \frac{5c^2r^2}{4}\left[11r^2 + 2de + 2\frac{r^3}{d} - \frac{er^2}{d} - 2re\right],$$

$$\prod_{i=1}^{i=5} x_i = \sqrt{\bar{c}}.$$

Applying σ_1 and σ_2 to $e = \dfrac{\rho_3^{\,2}}{\rho_4}$, we come to the *reductio ad absurdum* $e = 0$.

It follows then that $f(x) = 0$ is an irreducible Abelian equation under the derived conditions.

Type IV. If c, ω, d, r, e but not $\sqrt[5]{2^4ed^2r^2}$ and $\sqrt{c}$ are in the given rational domain, and if the identity exists

$$\frac{1}{c^2} = \frac{19}{4}r^5 + \frac{r^8}{16ed^2} + \frac{r^4e^2}{4d} - \frac{ed^2r^2}{2} + \frac{15}{8}\frac{r^6}{d} + \frac{15}{4}r^4e + \frac{5}{2}r^3de - \frac{5}{4}\frac{r^5e}{d},$$

then the above $f(x) = 0$ with the given roots is an irreducible Abelian equation of the tenth degree.

An example is furnished by supposing

$$r = d = e = 1, \qquad c = \frac{4}{\sqrt{183}}.$$

Here $\sqrt{183}$ must be in our rational domain as well as ω, but not $\sqrt[4]{183}$; further, the identity between r, d, e, c exists as given above and $2^4ed^2r^2$ is not a perfect fifth power; hence $\sqrt[5]{2}$ must be a quantity not in the given domain. The corresponding equation is

$$f(x) = x^{10} + \frac{619}{183}x^6 - \frac{1019,808}{183\sqrt{183}}x^4 + \frac{13714,88}{183^2}x^2 - \frac{4}{\sqrt{183}} = 0.$$

The roots are:

$$x_1 = -x_6 = \frac{2}{\sqrt[4]{183}}\left[-\sqrt[5]{\frac{1}{2}} + \sqrt[5]{\frac{1}{4}} + \sqrt[5]{\frac{1}{16}} + 1\right],$$

$$x_2 = -x_7 = \frac{2}{\sqrt[4]{183}}\left[-\omega^2\sqrt[5]{\frac{1}{2}} + \omega^4\sqrt[5]{\frac{1}{4}} + \omega^3\sqrt[5]{\frac{1}{16}} + 1\right],$$

$$\cdots \text{ etc. } \cdots$$

3. $\rho_i \neq 0$ $(i = 1 \text{ to } 4)$.

We proceed to show, that this case is impossible here just as case A. I, 2.

We saw above that, if λ_1 and λ_2 are in our given rational domain, we may put

$$\rho_1^2\rho_2 + \rho_4^2\rho_3 = \lambda_1 + \lambda_2\sqrt{5}, \tag{1}$$

$$\rho_2^2\rho_4 + \rho_3^2\rho_1 = \lambda_1 - \lambda_2\sqrt{5}, \tag{2}$$

$$\rho_1\rho_4 = +R + S\sqrt{5} \equiv T \neq 0, \tag{3}$$

$$\rho_2\rho_3 = +R - S\sqrt{5} \equiv Q \neq 0. \tag{4}$$

Eliminating ρ_2 and ρ_4 in (1) and (2) by means of (3) and (4), and finally removing ρ_3 in the new equation (2), we come to the result

$$\frac{A^2\rho_1^5}{4T^4} = \frac{\lambda_1 - \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 - \lambda_2\sqrt{5})^2 - 4Q^2T}}{2}, \tag{5}$$

where

$$A = \lambda_1 + \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 + \lambda_2\sqrt{5})^2 - 4T^2Q}.$$

Removing ρ_3 and ρ_4 in (1) and (2) by means of (3) and (4), and finally ρ_2, then equation (2) can be written

$$\frac{4\rho_1^5}{A^2} = \frac{\lambda_1 - \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 - \lambda_2\sqrt{5})^2 - 4Q^2T}}{2Q^2}. \tag{6}$$

Comparing (5) and (6) we have

$$A^2 = \pm 2^2QT^2. \tag{7}$$

The last relation may be written

$$(\lambda_1 + \lambda_2\sqrt{5})[\lambda_1 + \lambda_2\sqrt{5} \pm \sqrt{(\lambda_1 + \lambda_2\sqrt{5})^2 - 4QT^2}] = 4QT^2 \text{ or } = 0, \tag{8}$$

hence there are only two possibilities, namely

$$\lambda_1 + \lambda_2\sqrt{5} = 0, \tag{9}$$

or

$$(\lambda_1 + \lambda_2\sqrt{5})^2 = 4QT^2 \neq 0. \tag{10}$$

Doing the same thing with ρ_2, we find, that one of the following two relations must exist:

$$\lambda_1 - \lambda_2\sqrt{5} = 0 \tag{11}$$

or

$$(\lambda_1 - \lambda_2\sqrt{5})^2 = 4Q^2T. \tag{12}$$

Pairing equations (9) and (11), then $\lambda_1 = \lambda_2 = 0$; hence we get from

$$\text{equation (2) that } + \frac{1}{\rho_4^2} = -\frac{\rho_2^2}{\rho_3^2(2R - \rho_2\rho_3)},$$

$$\text{and equation (1) that } + \rho_4^4 = -\frac{\rho_2(2R - \rho_2\rho_3)^2}{\rho_3},$$

$$\therefore + 1 = \rho_4^4 \cdot \left(\frac{1}{\rho_4^2}\right)^2 = \frac{+\rho_2^5}{\rho_3^5}, \quad \text{or} \quad \rho_2 = \rho_3$$

and from (2) again we get, if $\rho_2 = \rho_3$, that $\rho_1 = -\rho_4$.

From $\rho_2 = \rho_3$ we have

$$N\sqrt{-10 + 2\sqrt{5}} - M\sqrt{-10 - 2\sqrt{5}} = 0,$$

and from $\rho_1 = -\rho_4$ we have

$$K + L\sqrt{5} = 0.$$

As now $\rho_i \neq 0$, it follows $\sqrt{5}$ is rationally known but then $\rho_1\rho_4 = -\rho_1^2 = -\rho_4^2$ as well as $\rho_2\rho_3 = \rho_2^2 = \rho_3^2$, and hence ρ_i is rationally known, so that $f(x)$ contains the rational factor $(x - x_1)(x - x_6)$. It follows then that $\lambda_1 = \lambda_2 = 0$ is excluded.

Pairing (9) and (12), then $\sqrt{5}$ is rationally known, as

$$\lambda_1 = -\lambda_2\sqrt{5} \neq 0$$

and we have from (12)

$$\lambda_1^2 = Q^2T,$$

or

$$\lambda_1 = (R - S\sqrt{5})\sqrt{R + S\sqrt{5}}.$$

From (2) we have

$$\rho_2^2\rho_4 = 2\lambda_1 - \rho_3^2\rho_1 = 2\lambda_1 - \frac{Q^2T}{\rho_2^2\rho_4},$$

or

$$(\rho_2^2\rho_4)^2 - 2\lambda_1\rho_2^2\rho_4 + \lambda_1^2 = 0,$$

$$\rho_2^2\rho_4 = \lambda_1.$$

It follows again by (2)

$$\rho_2^2\rho_4 = \rho_1\rho_3^2 = \lambda_1$$

and hence

$$\left.\begin{aligned} \rho_1 &= \frac{\rho_2^2\rho_4}{\rho_3^2}. && (13) \\ \rho_1^2\rho_2 &= -\rho_4^2\rho_3 && (14) \end{aligned}\right\} \text{eliminating } \rho_1$$

From (1) we have (14) above.

$$\therefore \frac{\rho_2^4\rho_4^2\rho_2}{\rho_3^4} = -\rho_4^2\rho_3,$$

or

$$\rho_2 = -\rho_3 \tag{15}$$

and hence $\rho_1 = \rho_4$ by (13). We have arrived at a similar impossible result as in the case $\lambda_1 = \lambda_2 = 0$.

The case of pairing (10) and (11) is identical to the case just discussed; the only difference lies in the other sign of $\sqrt{5}$.

It remains then to show that relation (10) and relation (12) cannot exist together. From (2) we have

$$\rho_2^2\rho_4 = \lambda_1 - \lambda_2\sqrt{5} - \rho_3^2\rho_1 = \lambda_1 - \lambda_2\sqrt{5} - \frac{Q^2T}{\rho_2^2\rho_4}$$

or

$$(\rho_2^2\rho_4)^2 - (\lambda_1 - \lambda_2\sqrt{5})\rho_2\rho_4 + Q^2T = 0,$$

$$\therefore \text{by (12)} \quad \rho_2^2\rho_4 = +\frac{\lambda_1 - \lambda_2\sqrt{5}}{2}$$

or by (2) again

$$\rho_2^2\rho_4 = \rho_3^2\rho_1 \quad \text{or} \quad \rho_1 = \frac{\rho_2^2\rho_4}{\rho_3^2}. \tag{16}$$

Similarly from (1) follows exactly in the same way by means of (10)

$$\rho_1^2\rho_2 = \frac{\lambda_1 + \lambda_2\sqrt{5}}{2} = \rho_4^2\rho_3. \tag{17}$$

Eliminating ρ_1 between (16) and (17), we are led to the relation

$$\rho_2 = \rho_3$$

and hence, by (16),

$$\rho_1 = \rho_4.$$

Now $\rho_1^2\rho_2 + \rho_2^2\rho_4 = \lambda_1$ and as $\rho_2 = \rho_3$

$$\rho_1^2\rho_3 + \rho_2^2\rho_4 = \lambda_1 = \text{rationally known.}$$

Applying however G_{10} to $\rho_1^2\rho_3 + \rho_2^2\rho_4$, we see it is changed into $\omega\rho_1^2\rho_3 + \rho_2^2\rho_4$ and, as $\rho_i \neq 0$, we have the result:

There is no Abelian equation corresponding to case A. II, 3.

B. ALL $\varphi_i = x_i + x_{i+5}$ ($i = 1$ TO 5) ARE DISTINCT.

The irreducible Abelian equations, left for consideration, do not reduce to quintics in x^2.

Let us consider first the five systems of imprimitivity by taking the function

$$\prod_{i=1}^{i=5}(x_i - x_{i+5}), \quad (i = 1 \text{ to } 5)$$

This function is a double-valued function under G_{10}; hence its square is a quantity in our given rational domain. Putting then

$$\gamma = \prod_{i=1}^{i=5}(x_i - x_{i+5})^2 \quad (i = 1 \text{ to } 5),$$

γ is rationally known, but not a perfect square.

The Lagrange resolvent

$$\psi_1 = x_1 + \epsilon x_7 + \epsilon^2 x_3 + \epsilon^3 x_9 + \epsilon^4 x_5 + \epsilon^5 x_6 + \epsilon^6 x_2 + \epsilon^7 x_8 + \epsilon^8 x_4 + \epsilon^9 x_{10}$$

can be written in the form

$$\psi_1 = x_1 - x_6 + \omega^3(x_2 - x_7) + \omega(x_3 - x_8) + \omega^4(x_4 - x_9) + \omega^2(x_5 - x_{10}).$$

Similarly we may write the other values of ψ_1 under G_{10} in the form:

$$\begin{aligned}
\psi_2 &= x_1 + x_6 + \omega(x_2 + x_7) + \omega^2(x_3 + x_8) + \omega^3(x_4 + x_9) + \omega^4(x_5 + x_{10}),\\
\psi_3 &= x_1 - x_6 + \omega^4(x_2 - x_7) + \omega^3(x_3 - x_8) + \omega^2(x_4 - x_9) + \omega(x_5 - x_{10}),\\
\psi_4 &= x_1 + x_6 + \omega^2(x_2 + x_7) + \omega^4(x_3 + x_8) + \omega(x_4 + x_9) + \omega^3(x_5 + x_{10}),\\
\psi_5 &= x_1 - x_6 + (x_2 - x_7) + (x_3 - x_8) + (x_4 - x_9) + (x_5 - x_{10}),\\
\psi_6 &= x_1 + x_6 + \omega^3(x_2 + x_7) + \omega(x_3 + x_8) + \omega^4(x_4 + x_9) + \omega^2(x_5 + x_{10}),\\
\psi_7 &= x_1 - x_6 + \omega(x_2 - x_7) + \omega^2(x_3 - x_8) + \omega^3(x_4 - x_9) + \omega^4(x_5 - x_{10}),\\
\psi_8 &= x_1 + x_6 + \omega^4(x_2 + x_7) + \omega^3(x_3 + x_8) + \omega^2(x_4 + x_9) + \omega(x_5 + x_{10}),\\
\psi_9 &= x_1 - x_6 + \omega^2(x_2 - x_7) + \omega^4(x_3 - x_8) + \omega(x_4 - x_9) + \omega^3(x_5 - x_{10}),\\
\psi_{10} &= x_1 + x_6 + (x_2 + x_7) + (x_3 + x_8) + x_4 + x_9) + (x_5 + x_{10}),
\end{aligned}$$

Taking our $f(x) = 0$ again in the reduced form, then

$$\sum_{j=1}^{j=10} x_j = \psi_{10} = 0 \quad \text{and} \quad \sum_{i=1}^{i=5} x_i = \frac{\psi_5}{2}.$$

Both functions, $\sum_{i=1}^{i=5} x_i$ and $\prod_{i=1}^{i=5}(x_i - x_{i+5})$, are double-valued functions under G_{10}; hence calling

$$5r = \frac{\sum_{i=1}^{i=5} x_i}{\prod_{i=1}^{i=5}(x_i - x_{i+5})},$$

then r *is in our given domain* and $\psi_5 = 10\sqrt{\gamma}\cdot r$.

All $\psi_a{}^5$ are unchanged by G_{10} if a is even; and if a is odd then $\frac{\psi_a{}^5}{\sqrt{c}}$ must be rationally known if we adjoin ω to the rational domain. Analogy to case A shows that we may write

$$\begin{aligned}
\psi_1 &= 10\sqrt{\gamma}\sqrt[5]{K_1 + L_1\sqrt{5} + M_1\sqrt{-10+2\sqrt{5}} + N_1\sqrt{-10-2\sqrt{5}}} = 10\rho_1\sqrt{\gamma},\\
\psi_2 &= 10\cdot\sqrt[5]{K_2 - L_2\sqrt{5} + N_2\sqrt{-10+2\sqrt{5}} - M_2\sqrt{-10-2\sqrt{5}}} = 10\rho_2,\\
\psi_3 &= 10\sqrt{\gamma}\sqrt[5]{K_1 - L_1\sqrt{5} - N_1\sqrt{-10+2\sqrt{5}} + M_1\sqrt{-10-2\sqrt{5}}} = 10\sqrt{\gamma}\rho_3,\\
\psi_4 &= 10\sqrt[5]{K_2 + L_2\sqrt{5} - M_2\sqrt{-10+2\sqrt{5}} - N_2\sqrt{-10-2\sqrt{5}}} = 10\rho_4,\\
\psi_6 &= 10\sqrt[5]{K_2 + L_2\sqrt{5} + M_2\sqrt{-10+2\sqrt{5}} + N_2\sqrt{-10-2\sqrt{5}}} = 10\rho_6,\\
\psi_7 &= 10\sqrt{\gamma}\sqrt[5]{K_1 - L_1\sqrt{5} + N_1\sqrt{-10+2\sqrt{5}} - M_1\sqrt{-10-2\sqrt{5}}} = 10\sqrt{\gamma}\rho_7,\\
\psi_8 &= 10\sqrt[5]{K_2 - L_2\sqrt{5} - N_2\sqrt{-10+2\sqrt{5}} + M_2\sqrt{-10-2\sqrt{5}}} = 10\rho_8,\\
\psi_9 &= 10\sqrt{\gamma}\sqrt[5]{K_1 + L\sqrt{5} - M_1\sqrt{-10+2\sqrt{5}} - N_1\sqrt{-10-2\sqrt{5}}} = 10\sqrt{\gamma}\rho_9,
\end{aligned}$$

$$\psi_5 = 5r\sqrt{\gamma}, \qquad \psi_{10} = 0.$$

As our $\varphi_1 = x_1 + x_6$ takes five distinct values under G_{10}, the quintic resolvent equation

$$\prod_{i=1}^{i=5} (\varphi - \varphi_i) = 0$$

is an irreducible Abelian equation of the fifth degree in φ and the corresponding group in the x's is G_{10}, which either leaves every φ_i fixed or interchanges them cyclically.

We have now from our Lagrange system by addition

$$\begin{aligned}
\varphi_1 &= x_1 + x_6 = 2[\rho_6 + \rho_8 + \rho_2 + \rho_4],\\
\varphi_2 &= x_2 + x_7 = 2[\omega^2\rho_6 + \omega\rho_8 + \omega^4\rho_2 + \omega^3\rho_4],\\
\varphi_3 &= x_3 + x_8 = 2[\omega^4\rho_6 + \omega^2\rho_8 + \omega^3\rho_2 + \omega\rho_4],\\
\varphi_4 &= x_4 + x_9 = 2[\omega\rho_6 + \omega^3\rho_8 + \omega^2\rho_2 + \omega^4\rho_4],\\
\varphi_5 &= x_5 + x_{10} = 2[\omega^3\rho_6 + \omega^4\rho_8 + \omega\rho_2 + \omega^2\rho_4],
\end{aligned}$$

and, defining

$$\varphi_{i+5} = x_i - x_{i+5} \qquad (i = 1 \text{ to } 5),$$

then

$$\begin{aligned}
\varphi_6 &= x_1 - x_6 = 2\sqrt{\gamma}[+\rho_1 + \rho_3 + \rho_7 + \rho_9 + r],\\
\varphi_7 &= x_2 - x_7 = 2\sqrt{\gamma}[\omega^2\rho_1 + \omega\rho_3 + \omega^4\rho_7 + \omega^3\rho_9 + r],\\
\varphi_8 &= x_3 - x_8 = 2\sqrt{\gamma}[\omega^4\rho_1 + \omega^2\rho_3 + \omega^3\rho_7 + \omega\rho_9 + r],\\
\varphi_9 &= x_4 - x_9 = 2\sqrt{\gamma}[\omega\rho_1 + \omega^3\rho_3 + \omega^2\rho_7 + \omega^4\rho_9 + r],\\
\varphi_{10} &= x_5 - x_{10} = 2\sqrt{\gamma}[\omega^3\rho_1 + \omega^4\rho_3 + \omega\rho_7 + \omega^2\rho_9 + r].
\end{aligned}$$

Knowing φ_i and φ_{i+5}, we know of course x_j ($j = 1$ to 10). The roots are:

$$\begin{aligned}
x_1, x_6 &= \rho_6 + \rho_8 + \rho_2 + \rho_4 \pm \sqrt{\gamma}[\rho_1 + \rho_3 + \rho_7 + \rho_9 + r],\\
x_2, x_7 &= \omega^2\rho_6 + \omega\rho_8 + \omega^4\rho_2 + \omega^3\rho_4 \pm \sqrt{\gamma}[\omega^2\rho_1 + \omega\rho_3 + \omega^4\rho_7 + \omega^3\rho_9 + r],\\
x_3, x_8 &= \omega^4\rho_6 + \omega^2\rho_8 + \omega^3\rho_2 + \omega\rho_4 \pm \sqrt{\gamma}[\omega^4\rho_1 + \omega^2\rho_3 + \omega^3\rho_7 + \omega\rho_9 + r],\\
x_4, x_9 &= \omega\rho_6 + \omega^3\rho_8 + \omega^2\rho_2 + \omega^4\rho_4 \pm \sqrt{\gamma}[\omega\rho_1 + \omega^3\rho_3 + \omega^2\rho_7 + \omega^4\rho_9 + r],\\
x_5, x_{10} &= \omega^3\rho_6 + \omega^4\rho_8 + \omega\rho_2 + \omega^2\rho_4 \pm \sqrt{\gamma}[\omega^3\rho_1 + \omega^4\rho_3 + \omega\rho_7 + \omega^2\rho_9 + r],
\end{aligned}$$

$$\left.\begin{array}{l}\text{where } +\sqrt{\gamma} \text{ belongs to } x_i\\ \text{and } -\sqrt{\gamma} \text{ belongs to } x_{i+5}\end{array}\right\} (i = 1 \text{ to } 5).$$

Actual computation shows

$$\psi_4\psi_6 = 10^2\sqrt[5]{P_2 + Q_2\sqrt{5}} = (\omega + \omega^4)F_1 + (\omega^2 + \omega^3)F_2 + \sum_{i=1}^{i=5}\varphi_i^2$$

and

$$\psi_2\psi_8 = 10^2\sqrt[5]{P_2 - Q_2\sqrt{5}} = (\omega^2 + \omega^3)F_1 + (\omega + \omega^4)F_2 + \sum_{i=1}^{i=5}\varphi_i^2,$$

where

$$\begin{aligned}
P_2 &= K_2^2 + 5L_2^2 + 10(M_2^2 + N_2^2),\\
Q_2 &= 2K_2L_2 - 2(M_2^2 + N_2^2) - 8M_2N_2,\\
F_1 &= \varphi_1\varphi_3 + \varphi_2\varphi_4 + \varphi_1\varphi_4 + \varphi_2\varphi_5 + \varphi_3\varphi_5,\\
F_2 &= \varphi_1\varphi_2 + \varphi_1\varphi_5 + \varphi_2\varphi_3 + \varphi_3\varphi_4 + \varphi_4\varphi_5.
\end{aligned}$$

Both F_1 and F_2 are unchanged by G_{10} and hence they are rationally known. We may therefore write

$$\psi_4\psi_6 = 100\rho_4\rho_6 = 100(R_2 + S_2\sqrt{5}),$$

$$\psi_2\psi_8 = 100\rho_2\rho_8 = 100(R_2 - S_2\sqrt{5}),$$

where R_2 and S_2 are defined by

$$P_2 = R^5 + 50R_2^3S_2 + 10R_2S^4,$$
$$Q_2 = 5R_2^4S_2 + 50R_2^2S_2^3 + 25S_2^5.$$

We have the relations:

$$\rho_4\rho_6 + \rho_2\rho_8 = 2R_2,$$

$$\rho_4\rho_6 - \rho_2\rho_8 = 2S_2\sqrt{5}.$$

We shall divide case B into (I.) $R_2 = 0$; (II.) $R_2 \neq 0$. The case $R_2 = 0$ will have two parts, (1) $S_2 = 0$; (2) $S_2 \neq 0$.

B. I. (1) $R_2 = S_2 = 0$.

Without loss of generality we may assume $\rho_6 = \rho_8 = 0$; hence we have here again the similar relation as under (A.)

$$2K_2 + \sqrt{-10 + 2\sqrt{5}}\left[M_2 - N_2 + (M_2 + N_2)\frac{4\sqrt{5}}{-10 + 2\sqrt{5}}\right] = 0.$$

Further from $\rho_6 = 0$ follows

$$\rho_4 = \sqrt[5]{2(K_2 + L_2\sqrt{5})}$$

and from $\rho_8 = 0$ follows

$$\rho_2 = \sqrt[5]{2(K_2 - L\sqrt{5})}.$$

It follows then $K_2 \neq 0$, for then $\rho_2 = -\rho_4$ and φ_1 would be zero; hence the expression in the bracket cannot vanish and therefore ω *is in our rational domain.*

For the same reason, at most three ρ_i can vanish and we shall divide the discussion into

$$(\alpha)\ \rho_2 = 0;\quad \rho_4 \neq 0 \qquad (\beta)\ \rho_2 \neq 0;\quad \rho_4 \neq 0.$$

(α) $\rho_2 = 0$; hence

$$K_2 = L\sqrt{5} \neq 0 \quad \text{and} \quad \rho_4 = \sqrt[5]{4K_2}$$

$4K_2$ is not a perfect fifth power, for then φ_1 would be rationally known. Putting

$$\frac{\psi_9}{\psi_4\sqrt{\gamma}} = \delta'; \qquad \frac{\psi_3}{\psi_4^2\sqrt{\gamma}} = \vartheta'; \qquad \frac{\psi_7}{\psi_4^3\sqrt{\gamma}} = \zeta'; \qquad \frac{\psi_1}{\psi_4^4\sqrt{\gamma}} = \eta'$$

then δ', ϑ', ζ', η' are quantities in our given domain, for numerators and denominators change at the same time the sign, or are left fixed by G_{10}. We get then

$$\varphi_6 = x_1 - x_6 = 2\sqrt{\gamma}[10^3\eta'\sqrt[5]{(4K_2)^4} + 10^2\zeta'\sqrt[5]{(4K_2)^3} + 10\vartheta'\sqrt[5]{(4K_2)^2} + \delta'\sqrt[5]{4K_2} + r].$$

Putting

$$10^3\eta' = \eta; \qquad 10\vartheta' = \vartheta; \qquad 10^2\zeta' = \zeta$$
$$\delta' = \delta$$

$4K_2 = \kappa$ is not a perfect fifth power, but rationally known
$\delta, \eta, \vartheta, \zeta$ are rationally known

we have

$$x_1, x_6 = \sqrt[5]{\kappa} \pm \sqrt{\gamma}[r + \delta\sqrt[5]{\kappa} + \vartheta\sqrt[5]{\kappa^2} + \zeta\sqrt[5]{\kappa^3} + \eta\sqrt[5]{\kappa^4}],$$
$$x_2, x_7 = \omega^3\sqrt[5]{\kappa} \pm \sqrt{\gamma}[r + \omega^3\delta\sqrt[5]{\kappa} + \omega\vartheta\sqrt[5]{\kappa^2} + \omega^4\zeta\sqrt[5]{\kappa^3} + \omega^2\eta\sqrt[5]{\kappa}],$$
$$\cdot \quad \cdot \quad \cdot \quad \text{etc.} \quad \cdot \quad \cdot \quad \cdot$$

The corresponding equation is

$$f(x) = [x^5 - A_1x^4 + A_2x^3 - A_3x^2 + A_4x - A_5] \times [x^5 - B_1x^4 + B_2x^3 - B_3x^2 + B_4x - B_5] = 0,$$

where

$$A_1, B_1 = \pm 5r\sqrt{\gamma},$$
$$A_2, B_2 = +10\gamma r^2 - 5\gamma\kappa(\delta\eta + \vartheta\zeta) \mp 5\eta\kappa\sqrt{\gamma},$$
$$A_3, B_3 = -15r\gamma\eta\kappa + 10\gamma\delta\zeta\kappa + 5\gamma\kappa\vartheta^2 \pm 5\sqrt{\gamma}[2\gamma r^3 - 3r\gamma\kappa(\eta\delta + \vartheta\zeta) + \zeta\kappa + \gamma\kappa^2\eta(\zeta^2 + \vartheta\eta) + \gamma\kappa\delta(\vartheta^2 + \delta\zeta)],$$
$$A_4, B_4 = 5\gamma^2r^4 + 5\gamma^2\kappa^2(\delta^2\eta^2 + \vartheta^2\zeta^2) - 5\gamma^2\kappa(\zeta^3\delta\kappa + \delta^3\vartheta + \eta^3\zeta\kappa^2 + \vartheta^3\eta\kappa) + 5\gamma\eta^2\kappa^2 - 15r^2\gamma^2(\delta\eta + \vartheta\zeta)\kappa + 10\gamma^2r\kappa(\zeta^2\eta\kappa + \delta^2\zeta + \eta^2\vartheta\kappa + \vartheta^2\delta) + 10\gamma r\zeta\kappa - 5\gamma^2\eta\delta\vartheta\zeta\kappa^2 - 15\gamma\delta\vartheta\kappa \pm 5\sqrt{\gamma}[-\gamma\zeta^3\kappa^2 - \vartheta\kappa - 3r^2\gamma\eta\kappa + 2\gamma r\vartheta^2\kappa - \gamma\eta\vartheta\zeta\kappa^2 - 3\gamma\delta^2\vartheta\kappa + 2\gamma\eta^2\delta\kappa^2 + 8r\gamma\delta\zeta\kappa],$$
$$A_5, B_5 = \kappa + 5\gamma^2\delta^4\kappa + 10\gamma\delta^2\kappa - 5\gamma^2\zeta\vartheta^3\kappa^2 - 5\gamma^2\eta^3\vartheta\kappa^3 - 5\gamma\zeta\eta\kappa^2 - 15\gamma^2\kappa^2\zeta\eta\delta^2 + 5\gamma^2\zeta^2\eta^2\kappa^3 - 5\gamma^2r^3\eta\kappa + 5\gamma^2r^2\vartheta^2\kappa + 10r^2\gamma^2\delta\zeta\kappa - 5r\gamma^2\zeta^3\kappa^2 - 5r\gamma\vartheta\kappa - 5r\gamma^2\eta\vartheta\zeta\kappa^2 - 15r\gamma\vartheta\delta\kappa - 15r\gamma^2\vartheta\delta^2\kappa + 10r\gamma^2\delta\eta^2\kappa^2 \pm \sqrt{\gamma}[\gamma^2\eta^5\kappa^4 + \gamma^2\vartheta^5\kappa^2 + \gamma^2\zeta^5\kappa^3 + \gamma^2\delta^5\kappa + \gamma^2r + 5\delta\kappa + 10\gamma\delta^3\kappa - 5\gamma^2\zeta\delta^3\eta\kappa^2 - 5\gamma^2\zeta\vartheta^3\delta\kappa^2 - 5\gamma^2\eta\zeta^3\vartheta\kappa^3 - 5\gamma^2\delta\eta^3\vartheta\kappa^3 - 15\gamma\eta\zeta\delta\kappa^2 + 5\gamma^2\kappa^2(\zeta^2\eta^2\delta\kappa + \zeta^2\vartheta\delta^2 + \eta^2\zeta\vartheta^2\kappa + \delta^2\eta\vartheta^2) + 5\gamma\zeta^2\vartheta\kappa^2 + 5\gamma\eta\vartheta^2\kappa^2 - 5r^3\gamma^2\kappa(\zeta\vartheta + \eta\delta) + 5\gamma^2\kappa r^2(\zeta^2\eta\kappa + \delta^2\zeta + \eta^2\vartheta\kappa + \vartheta^2\delta) + 5\gamma r^2\zeta\kappa - 5r\gamma^2\kappa(\zeta^3\delta\kappa + \delta^3\vartheta + \eta^3\zeta\kappa^2 + \vartheta^3\eta\kappa) + 5r\gamma^2\kappa^2(\eta^2\delta^2 + \zeta^2\vartheta^2) - 5\gamma^2r\eta\delta\zeta\vartheta\kappa^2 + 5r\gamma\eta^2\kappa^2].$$

$\gamma, r, \delta, \eta, \vartheta, \zeta$ are not independent of each other, for the condition

$$\prod_{i=1}^{i=5}(x_i - x_{i+5}) = \sqrt{\gamma}$$

leads to *the identity*

$$\frac{1}{2^5\gamma^2} = \eta^5\kappa^4 + \zeta^5\kappa^3 + \vartheta^5\kappa^2 + \delta^5\kappa + r^5 - 5[\eta^3\vartheta\delta\kappa^3 + \vartheta^3\zeta\delta\kappa^2 + \zeta^3\eta\vartheta\kappa^3 + \delta^3\eta\zeta\kappa^2] + 5[\eta\vartheta^2\delta^2\kappa^2 + \vartheta\zeta^2\delta^2\kappa^2 + \zeta\eta^2\vartheta^2\kappa^3 + \delta\eta^2\zeta^2\kappa^3] + 5r\kappa^2(\eta^2\delta^2 + \vartheta^2\zeta^2) + 5r^2(\eta^2\vartheta\kappa^2 + \vartheta^2\delta\kappa + \zeta^2\eta\kappa^2 + \delta^2\zeta\kappa) - 5r^3(\delta\eta + \vartheta\zeta)\kappa - 5r(\vartheta^3\zeta\kappa^3 + \vartheta^3\eta\kappa^2 + \zeta^3\delta\kappa^2 + \delta^3\vartheta\kappa) - 5\kappa^2\delta\eta\zeta\vartheta.$$

There is obviously another condition to be imposed upon r, δ, ζ, ϑ, η namely *not all five can vanish* at once, for if

$$r = \delta = \zeta = \vartheta = \eta,$$

then the irrationality $\sqrt{\gamma}$ disappears entirely and $f(x)$ splits up into the two rational factors

$$\prod_{i=1}^{i=5}(x - x_i) \quad \text{and} \quad \prod_{i=1}^{i=5}(x - x_{i+5}),$$

as can be seen from the above A_i and B_i.

We ask again: Are the derived conditions also sufficient, that the above $f(x) = 0$ is an irreducible Abelian equation?

$f(x)$ cannot contain a linear, rationally known factor, for $\sqrt{\gamma}$ and $\sqrt[5]{\kappa}$ cannot be expressed rationally by each other; the former satisfies an irreducible quadratic, the latter an irreducible quintic. The sum of any two, three, or four roots is of the forms respectively

$$(\omega^\lambda + \omega^\mu)\sqrt[5]{\kappa} + \sqrt{\gamma}f_1(\omega, \sqrt[5]{\kappa}),$$

$$(\omega^\lambda + \omega^\mu + \omega^\nu)\sqrt[5]{\kappa} + \sqrt{\gamma}f_2(\omega, \sqrt[5]{\kappa}),$$

$$(\omega^\lambda + \omega^\mu + \omega^\nu + \omega^\pi)\sqrt[5]{\kappa^3} + \sqrt{\gamma}f_2(\omega, \sqrt[5]{\kappa}).$$

Hence, for the same reason that $\sqrt{\gamma}$ and $\sqrt[5]{\kappa}$ cannot be expressed rationally by each other, the last three forms must be irrational, even if

$$f_i(\omega, \sqrt[5]{\kappa}) = 0$$

$\sqrt[5]{\kappa}$ can disappear in a factor of the fifth degree, only if the roots of this factor are x_i or x_{i+5} ($i = 1$ to 5) and in fact the above symmetric functions of x_i and x_{i+5}, which we called A_i and B_i, show $\sqrt{\gamma}$ as the only irrationality. Our question is then simply: Can $\sqrt{\gamma}$ disappear in all A_i ($i = 1$ to 5) without contradicting the derived necessary conditions? A_1 and A_2 show that we must have (*a*) $r = \eta = 0$. Then we must have further, if $\sqrt{\gamma}$ shall disappear,

$$(b)\ \ \zeta + \gamma\zeta\delta^2 + \gamma\delta\vartheta^2 = 0 \quad (\text{by } A_3)$$

and

$$(c)\ \ \vartheta + 3\gamma\vartheta\delta^2 + \gamma\zeta^3\kappa = 0 \quad (\text{by } A_4).$$

Let us first see whether one of the quantities ζ, ϑ, δ, $1 + 3\gamma\delta^2$, can vanish without contradicting the necessary conditions derived.

(1) If $\delta = 0$ then by (*b*) $\zeta = 0$ and hence by (*c*) $\vartheta = 0$.

This case would require

$$r = \eta = \delta = \zeta = \vartheta = 0$$

and contradicts our conditions.

(2) If $\zeta = 0$ then by (*b*) $\delta = 0$ or $\vartheta = 0$ as $\gamma \neq 0$.

$\zeta = \delta = 0$ requires $\vartheta = 0$ by means of (*c*) and is therefore again impossible.

If then

$$r = \eta = \zeta = \vartheta = 0 \qquad \delta \neq 0$$

then A_5 shows that

$$\gamma^2\delta^4 + 10\gamma\delta^2 + 5 = 0$$

if $\sqrt{\gamma}$ shall disappear in A_5. Hence

$$\gamma\delta^2 \neq -5 \pm 2\sqrt{5}$$

and, as our identity leads to

$$\frac{1}{2^5\gamma^2} = \delta^5\kappa,$$

we must have

γ is not a perfect fifth power

for otherwise κ is a perfect fifth power which contradicts the conditions.

(3) If $\vartheta = 0$ then $\zeta = 0$, which is again case (2).

(4) If $1 + 3\gamma\delta^2 = 0$ then $\zeta = 0$, which is again case (2).

Assuming then that (*a*), (*b*) and (*c*) are true and that $\zeta \neq 0$, $\vartheta \neq 0$, $\delta \neq 0$, $1 + 3\gamma\delta^2 \neq 0$, we must have

$$\vartheta = -\frac{\gamma\zeta^3\kappa}{1 + 3\gamma\delta^2} \text{ by } (c)$$

and hence by (*b*)

$$\zeta^5 = -\frac{(1 + \gamma\delta^2)(1 + 3\gamma\delta^2)^2}{\gamma^3\delta\kappa^2}$$

so that from A_5 it must follow

$$1 + 15\gamma\delta^2 + 60\gamma^2\delta^4 + 75\gamma^3\delta^6 + 25\gamma^4\delta^8 \neq 0.$$

In this last case, that is

$$r = \eta = 0, \qquad \vartheta = -\frac{\gamma\zeta^3\kappa}{1 + 3\gamma\delta^2}, \qquad \zeta^5 = \frac{(1 + \gamma\delta^2)(1 + 3\gamma\delta^2)^2}{\gamma^3\delta\kappa^2}$$

our identity reduces to

$$\kappa = \frac{\gamma^2\delta^3}{2^5[1 + \gamma\delta^2 + 14\gamma^2\delta^4 + 48\gamma^3\delta^6 + 20\gamma^4\delta^8]}.$$

Under these conditions $\prod_{i=1}^{i=5}(x - x_i)$ must contain coefficients involving $\sqrt{\gamma}$.

This proves, that $f(x)$ is irreducible, if all conditions are observed. Furthermore the group of $f(x) = 0$ must have two as well as five systems of intransitivity; for A_i and B_i show that $f(x)$ splits up into two rational factors after having adjoined $\sqrt{\gamma}$ to the rational domain; and having adjoined ρ_4 to the rational domain then $f(x) = 0$ splits up into five equations with rational coefficients, namely into the five equations

$$(x - x_i)(x - x_{i+5}) = 0.$$

It follows then that the group of $f(x) = 0$ is a subgroup of the G_{240} above and it must be G_{10}, as σ_1 as well as σ_2 change $\kappa = \psi_4{}^5 \neq 0$ into a quantity which is zero.

Type V. If

$$\omega,\ r,\ \gamma,\ \delta,\ \vartheta,\ \zeta,\ \eta,\ \kappa$$

are quantities of the given rational domain, but not

$$\sqrt{\gamma}, \quad \sqrt[5]{\kappa}$$

and if the identity exists as given on page 141 *then* $f(x) = 0$ *with the roots on page* 141 *is an irreducible Abelian equation, if not all of the five quantities* r, ϑ, η, ζ, δ *are zero—*

with two special exceptions, namely

(1) *If* $r = \eta = \vartheta = \zeta = 0$ *then* $\gamma\delta^2 \neq -5 \pm 2\sqrt{5}$, $[\because \sqrt[5]{\gamma} = \textit{irrational}]$

(2) *If* $r = \eta = 0, \quad \zeta^5 = -\dfrac{(1+\gamma\delta^2)(1+3\gamma\delta^2)^2}{\gamma^3\delta\kappa^2}, \quad \vartheta = -\dfrac{\gamma\zeta^3\kappa}{1+3\gamma\delta^2}$

then

$$1 + 15\gamma\delta^2 + 60\gamma^2\delta^4 + 75\gamma^3\delta^6 + 25\gamma^4\delta^8 \neq 0.$$

Illustrations:

(1) $$\gamma = 2, \quad \eta = 2, \quad r = \vartheta = \zeta = \delta = 0;$$

then from the identity we have $\kappa = \dfrac{1}{8}$.

Hence the roots are:

$$x_1, x_6 = \sqrt[5]{\frac{1}{8}} \pm \sqrt{2}\sqrt[5]{\frac{1}{8^4}},$$

$$x_2, x_7 = \omega^3\sqrt[5]{\frac{1}{8}} \pm \omega^2\sqrt{2}\sqrt[5]{\frac{1}{8^4}},$$

$$\cdot \quad \cdot \quad \cdot \quad \text{etc.} \quad \cdot \quad \cdot \quad \cdot$$

The corresponding equation is

$$f(x) = 16x^{10} - 60x^6 - 4x^5 - 20x^3 + 25x^2 - 5x - 0,75 = 0.$$

Our domain must include $R(\omega)$.

(2) $$\zeta = \vartheta = \delta = \eta = 0, \quad \kappa = 2, \quad r = 2$$

then by the identity we get

$$\gamma = 2^{-5}.$$

Our domain must include $R(\omega)$.

$$f(x) = x^{10} + \frac{55}{2^5}x^6 - 4x^5 + \frac{45}{2^8}x^4 + \frac{5}{2^{12}}x^2 - \frac{5}{2^4}x + \frac{2^{10}-1}{2^{15}} = 0.$$

The roots are:

$$x_1, x_5 = \sqrt[5]{2} \pm \sqrt{\frac{1}{2^3}},$$

$$x_2, x_7 = \omega^3\sqrt[5]{2} \pm \sqrt{\frac{1}{2^3}}.$$

$$\cdot \quad \cdot \quad \cdot \quad \text{etc.} \quad \cdot \quad \cdot \quad \cdot$$

(β) $\rho_2 \neq 0$, $\rho_4 \neq 0$.

As ω is rationally known, we may put

$$\rho_2{}^5 = \lambda, \quad \rho_4{}^5 = \kappa$$

where λ, κ *are in our given domain.* Similarly, as under (α), we may write, if ν', μ', δ', ϑ' are in the given domain

$$\frac{\psi_7}{\sqrt{\gamma}\psi_2} = \nu', \qquad \frac{\psi_9}{\sqrt{\gamma}\psi_4} = \delta',$$

$$\frac{\psi_1}{\sqrt{\gamma}\psi_2{}^3} = \mu', \qquad \frac{\psi_3}{\sqrt{\gamma}\psi_4{}^2} = \vartheta'.$$

Hence

$$\varphi_6 = x_1 - x_6 = 2\sqrt{\gamma}[10^2\mu'\sqrt[5]{\lambda^3} + 10\vartheta'\sqrt[5]{\kappa^2} + \nu'\sqrt[5]{\lambda} + \delta'\sqrt[5]{\kappa} + r]$$

or, putting

$$10^2\mu' = \mu; \quad 10\vartheta' = \vartheta; \quad \nu' = \nu; \quad \delta' = \delta,$$

then ϑ, δ, μ, ν *are rationally known* and *not all five quantities*, ϑ, δ, μ, ν, r *can vanish*, as then $\varphi_6 = 0$. Our roots become

$$x_1,\, x_6 = \sqrt[5]{\lambda} + \sqrt[5]{\kappa} \pm \sqrt{\gamma}[\mu\sqrt[5]{\lambda^3} + \vartheta\sqrt[5]{\kappa^2} + \nu\sqrt[5]{\lambda} + \delta\sqrt[5]{\kappa} + r],$$

$$x_2,\, x_7 = \omega^4\sqrt[5]{\lambda} + \omega^3\sqrt[5]{\kappa} \pm \sqrt{\gamma}[\omega^2\mu\sqrt[5]{\lambda^3} + \omega\vartheta\sqrt[5]{\kappa^2} + \omega^4\nu\sqrt[5]{\lambda} + \omega^3\delta\sqrt[5]{\kappa} + r].$$

$$\cdot\ \cdot\ \cdot\ \text{etc.}\ \cdot\ \cdot\ \cdot$$

Both $\sqrt[5]{\lambda}$ *and* $\sqrt[5]{\kappa}$ *are not in our given domain*, for otherwise $(x - x_i)(x - x_{i+5})$ is a rational factor of $f(x)$ and as both $\sqrt[5]{\lambda}$ and $\sqrt[5]{\kappa}$ are rational or irrational at the same time. Further, $\lambda \neq \pm\kappa$, for then

$$\nu'\vartheta' = \frac{\psi_3\psi_7}{\gamma\psi_2\psi_4^2}$$

would become

$$\frac{\psi_3\psi_7}{\gamma\psi_2^3} = \nu'\vartheta'.$$

Applying G_{10} to $\psi_3\psi_7$ we see that it is a rationally known function and hence ψ_2^3 and therefore $\rho_2 = \sqrt[5]{\lambda}$ would be rational.

$\psi_2\psi_4^2$ is unchanged by G_{10}; hence if $\tau = \sqrt[5]{\lambda\kappa^2}$ then τ *is a quantity in our rational domain* and we have the relations:

$$\lambda = \frac{\tau^5}{\kappa^2}; \qquad \sqrt[5]{\lambda^3\kappa} = \frac{\tau^3}{\kappa}; \qquad \sqrt[5]{\lambda^4\kappa^3} = \frac{\tau^4}{\kappa}.$$

The quantities

$$\tau,\ \kappa,\ \gamma,\ r,\ \vartheta,\ \delta,\ \mu,\ \nu$$

are connected by *the identity*, based upon

$$\prod_{i=1}^{i=5}(x_i - x_{i+5}) = \sqrt{\gamma}$$

$$\begin{aligned}\frac{1}{2^5\gamma^2} = {}& \frac{\mu^5\tau^{15}}{\kappa^6} + \vartheta^5\kappa^2 + \frac{\nu^5\tau^5}{\kappa^2} + \delta^5\kappa + r^5 - 5\left[\mu^3\vartheta\delta\frac{\tau^9}{\kappa^3} + \vartheta^3\nu\delta\kappa\tau + \nu^3\mu\vartheta\frac{\tau^6}{\kappa^2} + \delta^3\mu\nu\frac{\tau^4}{\kappa}\right]\\ &+ 5\left[\mu\vartheta^2\delta^2\tau^3 + \vartheta\nu^2\delta^2\tau^2 + \nu\mu^2\vartheta^2\frac{\tau^7}{\kappa^2} + \delta\mu^2\nu^2\frac{\tau^8}{\kappa^3}\right] + 5r\left[\mu^2\delta^2\frac{\tau^6}{\kappa^2} + \vartheta^2\nu^2\tau^2\right]\\ &+ 5r\left[\mu^2\vartheta\frac{\tau^6}{\kappa^2} + \vartheta^2\delta\kappa + \nu^2\mu\frac{\tau^5}{\kappa^2} + \delta^2\nu\tau\right] - 5r\left[\mu^3\nu\frac{\tau^{10}}{\kappa^4} + \vartheta^3\mu\tau^3 + \nu^3\delta\frac{\tau^3}{\kappa} + \delta^3\vartheta\kappa\right]\\ &- 5r^3\left[\mu\delta\frac{\tau^3}{\kappa} + \vartheta\nu\tau\right] - 5\mu\nu\delta\vartheta r\frac{\tau^4}{\kappa}.\end{aligned}$$

The roots may now be written in the form

$$x_1,\, x_6 = \frac{\tau}{\sqrt[5]{\kappa^2}} + \sqrt[5]{\kappa} \pm \sqrt{\gamma}\left[\frac{\mu\tau^3}{\kappa\sqrt[5]{\kappa}} + \vartheta\sqrt[5]{\kappa^2} + \frac{\nu\tau}{\sqrt[5]{\kappa^2}} + \delta\sqrt[5]{\kappa} + r\right],$$

$$x_2,\, x_7 = \frac{\omega^4\tau}{\sqrt[5]{\kappa^2}} + \omega^3\sqrt[5]{\kappa} \pm \sqrt{\gamma}\left[\frac{\omega^2\mu\tau^3}{\kappa\sqrt[5]{\kappa}} + \omega\vartheta\sqrt[5]{\kappa^2} + \frac{\omega^4\nu\tau}{\sqrt[5]{\kappa^2}} + \omega^3\delta\sqrt[5]{\kappa} + r\right],$$

$$\cdot\ \cdot\ \cdot\ \text{etc.}\ \cdot\ \cdot\ \cdot$$

The corresponding equation is

$$
\begin{aligned}
f(x) = (x^5 - A_1x^4 + A_2x^3 - A_3x^2 + A_4x - A_5)& \\
\times (x^5 - B_1x^4 + B_2x^3 - B_3x^2 + B_4x - B_5) &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1, B_1 &= \pm 5r\sqrt{\gamma}, \\
A_2, B_2 &= 10r^2\gamma - 5\gamma\tau\biggl(\mu\delta\frac{\tau^2}{\kappa} + \vartheta\nu\biggr) \mp 5\tau\sqrt{\gamma}\biggl(\mu\frac{\tau^2}{\kappa} + \vartheta\biggr), \\
A_3, B_3 &= 5\tau + 5\gamma(\delta^2\tau + \vartheta^2\kappa) - 15r\gamma\tau\biggl(\mu\frac{\tau^2}{\kappa} + \vartheta\biggr) + 10\gamma\biggl(\mu\nu\frac{\tau^5}{\kappa^2} + \nu\delta\tau\biggr) \\
&\quad \pm 5\sqrt{\gamma}\biggl[2\gamma r^3 + \mu\frac{\tau^5}{\kappa^2} + \nu\tau + 2\delta\tau + \nu^2\mu\gamma\frac{\tau^5}{\kappa^2} + \gamma\delta^2\nu\tau + \gamma\mu^2\vartheta\frac{\tau^6}{\kappa^2} + \vartheta^2\delta\kappa \\
&\quad - 3\gamma r\mu\delta\frac{\tau^3}{\kappa} - 3\gamma r\vartheta\nu\tau\biggr], \\
A_4, B_4 &= -5\frac{\tau^3}{\kappa} + 5\gamma^2r^4 + 5\gamma^2\tau^2\biggl(\mu^2\delta^2\frac{\tau^4}{\kappa^2} + \vartheta^2\nu^2\biggr) - 5\gamma^2\biggl(\nu^3\delta\frac{\tau^3}{\kappa} + \delta^3\vartheta\kappa \\
&\quad + \mu^3\nu\frac{\tau^{10}}{\kappa^4} + \vartheta^3\mu\tau^3\biggr) + 5\gamma\tau^2(\mu^2\tau + \vartheta^2) - 15r^2\gamma^2\tau\biggl(\mu\delta\frac{\tau^2}{\kappa} + \vartheta\nu\biggr) \\
&\quad + 10\gamma^2r\biggl(\nu^2\mu\frac{\tau^5}{\kappa^2} + \delta^2\nu\tau + \mu^2\vartheta\frac{\tau^6}{\kappa^2} + \vartheta^2\delta\kappa\biggr) + 10\gamma r\biggl(\mu\frac{\tau^5}{\kappa^2} + \nu\tau\biggr) \\
&\quad - 5\gamma^2\mu\vartheta\delta\nu\frac{\tau^4}{\kappa} - 5\gamma\mu\vartheta\frac{\tau^4}{\kappa} - 15\gamma\nu\frac{\tau^3}{\kappa}(\delta + \nu) - 15\gamma\delta\vartheta\kappa \\
&\quad + 40r\gamma\delta\tau \pm 5\sqrt{\gamma}\biggl[2r\tau - \gamma\tau^3\biggl(\frac{\nu^3}{\kappa} + \frac{\mu^3\tau^7}{\kappa^4}\biggr) - \frac{\delta\tau^3}{\kappa} - \vartheta\kappa \\
&\quad - 3r^2\gamma\tau\biggl(\mu\frac{\tau^2}{\kappa} + \vartheta\biggr) + 2\gamma r(\delta^2\tau + \vartheta^2\kappa) - \gamma\vartheta\mu\frac{\tau^4}{\kappa}(\nu + \delta) - 3\gamma\nu^2\delta\frac{\tau^3}{\kappa} \\
&\quad - 3\nu\frac{\tau^3}{\kappa} - 3\gamma\delta^2\vartheta\kappa + 2\gamma\mu^2\delta\frac{\tau^6}{\kappa^2} + 2\gamma\vartheta^2\nu\tau^2 + 8r\gamma\nu\tau\biggl(\mu\frac{\tau^4}{\kappa^2} + \delta\biggr)\biggr], \\
A_5, B_5 &= \frac{\tau^5}{\kappa^2} + \kappa + 5\gamma^2\biggl(\frac{\nu^4\tau^5}{\kappa^2} + \delta^4\kappa\biggr) + 10\gamma\biggl(\frac{\nu^2\tau^5}{\kappa^2} + \delta^2\kappa\biggr) \\
&\quad - 5\gamma^2\nu\vartheta^3\kappa\tau - 5\gamma^2\tau\biggl(\delta^3\mu\frac{\tau^3}{\kappa} + \vartheta^3\delta\kappa + \mu^3\vartheta\frac{\tau^8}{\kappa^3}\biggr) \\
&\quad - 5\gamma\mu\tau\biggl(\nu\frac{\tau^3}{\kappa} + \vartheta\frac{\tau^5}{\kappa^2}\biggr) - 15\gamma\mu\delta\frac{\tau^4}{\kappa} - 15\gamma^2\mu\nu\tau\biggl(\delta^2\frac{\tau^3}{\kappa} + \vartheta\nu\frac{\tau^5}{\kappa^2}\biggr) \\
&\quad + 5\gamma\mu^2\frac{\tau^9}{\kappa^3} + 5\gamma^2\mu^2\frac{\tau^7}{\kappa^2}\biggl(\nu^2\frac{\tau}{\kappa} + \vartheta^2\biggr) + 10\gamma\vartheta\tau^2(\delta + \nu) \\
&\quad - 5\gamma^2r^3\tau\biggl(\vartheta + \mu\frac{\tau^2}{\kappa}\biggr) + 5\gamma r^2\tau + 5\gamma^2r^2(\delta^2\tau + \vartheta^2\kappa) \\
&\quad + 10r^2\gamma^2\nu\tau\biggl(\mu\frac{\tau^4}{\kappa^2} + \delta\biggr) - 5r\gamma^2\tau^3\biggl(\frac{\nu^3}{\kappa} + \mu^3\frac{\tau^7}{\kappa^4}\biggr) - 5r\gamma\biggl(\delta\frac{\tau^3}{\kappa} + \vartheta\kappa\biggr) \\
&\quad - 5r\gamma^2\mu\vartheta\frac{\tau^4}{\kappa}(\nu + \delta) - 15r\gamma\nu\frac{\tau^3}{\kappa} - 15r\gamma\delta\biggl(\nu\frac{\tau^3}{\kappa} + \vartheta\kappa\biggr) \\
&\quad - 15r\gamma^2\delta\biggl(\nu^2\frac{\tau^3}{\kappa} + \vartheta\delta\kappa\biggr) + 10r\gamma^2\tau^2\biggl(\nu\vartheta^2 + \delta\mu^2\frac{\tau^4}{\kappa^2}\biggr) \\
&\quad \pm \sqrt{\gamma}\biggl[5\nu\frac{\tau^5}{\kappa^2} + 5\delta\kappa + \gamma^2\biggl(\mu^5\frac{\tau^{15}}{\kappa^6} + \vartheta^5\kappa^2 + \frac{\nu^5\tau^5}{\kappa^2} + \delta^5\kappa + r\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+ 10\gamma\left(\frac{\nu^3\tau^5}{\kappa^2} + \delta^3\kappa\right) - 5\gamma^2\tau\left(\mu\nu\delta^3\frac{\tau^3}{\kappa} + \nu\vartheta^3\delta\kappa + \mu\nu^3\vartheta\frac{\tau^5}{\kappa^2} + \delta\mu^3\vartheta\frac{\tau^8}{\kappa}\right) \\
&- 5\gamma\vartheta^3\kappa\tau - 5\mu\frac{\tau^4}{\kappa} - 15\gamma\mu\nu\tau\left(\delta\frac{\tau^3}{\kappa} + \vartheta\frac{\tau^5}{\kappa^2}\right) - 15\gamma\mu\delta^2\frac{\tau^4}{\kappa} \\
&- 5\gamma^2\tau^2\left(\nu^2\mu^2\delta\frac{\tau^6}{\kappa^3} + \nu^2\vartheta\delta^2 + \mu^2\vartheta^2\nu\frac{\tau^5}{\kappa^2} + \delta^2\vartheta^2\mu\tau\right) + 5\vartheta\tau^2(1 + \gamma\nu^2) \\
&+ 5\gamma\tau^2\left(\delta\mu^2\frac{\tau^6}{\kappa} + \vartheta\delta^2 + \mu\vartheta^2\tau\right) + 20\gamma\vartheta\nu\delta\tau^2 - 5\gamma^2r^3\tau\left(\nu\vartheta + \mu\delta\frac{\tau^2}{\kappa}\right) \\
&+ 5\gamma^2r^2\left(\nu^2\mu\frac{\tau^5}{\kappa^2} + \delta^2\nu\tau + \mu^2\vartheta\frac{\tau^6}{\kappa^2} + \vartheta^2\delta\kappa\right) + 5\gamma r^2\tau\left(\mu\frac{\tau^4}{\kappa^2} + \nu\right) \\
&+ 10r^2\gamma\delta\tau - 5r\frac{\tau^3}{\kappa} - 5r\gamma^2\left(\nu^3\delta\frac{\tau^3}{\kappa} + \delta^3\vartheta\kappa + \mu^3\nu\frac{\tau^{10}}{\kappa^4} + \vartheta^3\mu\tau^3\right) \\
&+ 5r\gamma^2\tau^2\left(\nu^2\vartheta^2 + \delta^2\mu^2\frac{\tau^4}{\kappa^2}\right) - 5r\gamma^2\mu\vartheta\nu\delta\frac{\tau^4}{\kappa} + 5r\gamma\tau^2\left(\vartheta^2 + \mu^2\frac{\tau^4}{\kappa^2}\right) \\
&- 5r\gamma\mu\vartheta\frac{\tau^4}{\kappa} - 15r\gamma\nu^2\frac{\tau^3}{\kappa}\Bigg].
\end{aligned}
$$

Regarding the sufficiency of the derived conditions, we first note that the group of the above $f(x) = 0$ must contain two as well as five systems of intransitivity; for, adjoining $\sqrt[5]{\kappa}$ to the rational domain, $f(x)$ splits up into the five rational factors $(x - x_i)(x - x_{i+5})$, as seen directly from the roots of $f(x) = 0$. The above A_i and B_i show, that $f(x)$ is the product of two rational factors, if $\sqrt{\gamma}$ is adjoint to the rational domain. It follows then that the group of $f(x) = 0$ must be a subgroup of the above G_{240}, if $f(x)$ is irreducible.

Just as under (α) the sum of any two, three, or four roots can not be rational under the derived conditions, as $\sqrt{\gamma}$ and $\sqrt[5]{\kappa}$ cannot be expressed rationally by each other and as ω is rationally known. The same is true for any single root and hence $f(x)$ cannot contain any rational factor of the first, second, third or fourth degree. The sum of any five roots is either irrational for the same reason or it is zero, which can happen, only when we pick out the five roots belonging to the same system of intransitivity. It remains then to see when the coefficients of $\sqrt{\gamma}$ in all A_i may vanish. A_1 and A_2 can be rational only if respectively

$$(a)\quad r = 0 \qquad\qquad (b)\quad \mu\frac{\tau^2}{\kappa} + \vartheta = 0$$

as $\tau \neq 0$, $\gamma \neq 0$.

A_3 is rational under the assumption that (a) and (b) hold and if

$$(c)\qquad -\frac{\vartheta\tau^2}{\kappa} + \nu + 2\delta + \gamma\left(-\frac{\nu^2\vartheta\tau^2}{\kappa} + \delta^2\nu + \vartheta^3 + \frac{\vartheta^2\delta\kappa}{\tau}\right) = 0$$

Assuming the truth of (a) and (b), again A_4 is rational, if

$$
\begin{aligned}
(d)\qquad &+ \gamma\left(\frac{\vartheta^3\tau^2}{\kappa} - \nu^3\tau\right) - \left(\delta\tau + \frac{\vartheta\kappa}{\tau^2}\right) + \gamma\vartheta^2\kappa(\nu + \delta) - 3\gamma\nu^2\delta\tau - 3\nu\tau \\
&\qquad - \frac{3\gamma\kappa\vartheta\delta^2}{\tau^2} + 2\gamma\delta\vartheta^2 + 2\gamma\nu\vartheta^2 = 0,
\end{aligned}
$$

and A_5 is rational, if

$$\frac{1}{5}\gamma^2\left(-\frac{\vartheta^5\tau^3}{\kappa}+\frac{\vartheta^5\kappa^2}{\tau^2}+\frac{\nu^5\tau^3}{\kappa^2}+\frac{\delta^5\kappa}{\tau^2}\right)+\frac{\nu\tau^3}{\kappa^2}+\frac{\delta\kappa}{\tau^2}+2\gamma\left(\frac{\nu^3\tau^3}{\kappa^2}+\frac{\delta^3\kappa}{\tau^2}\right)$$

$$(e)\qquad -\gamma^2\vartheta\left(\frac{\nu\delta\vartheta^2\kappa}{\tau}-\nu\delta^3-\nu^3\vartheta\tau-\vartheta^2\delta\tau\right)-\frac{\gamma\vartheta^3\kappa}{\tau}+2\vartheta+3\gamma\vartheta\nu\left(\delta+\frac{\vartheta\tau^2}{\kappa}\right)$$

$$+3\gamma\vartheta\delta^2\tau-\gamma^2\vartheta\left(\frac{\nu^2\vartheta\delta\tau^2}{\kappa^2}+\nu^2\delta^2+\nu\vartheta^3\tau-\frac{\delta^2\vartheta^2\kappa}{\tau}\right)+\vartheta\gamma\nu^2$$

$$+\gamma\vartheta\left(\frac{\delta\vartheta\tau^2}{\kappa}+\delta^2-\frac{\vartheta^2\kappa}{\tau}\right)+20\vartheta\nu\delta=0.$$

We have then the following condition: if $r \neq 0$, $f(x)$ is irreducible and *if* $r = 0$ *and* $\mu = -\vartheta\kappa/\tau^2$ *then* $\tau, \kappa, \nu, \delta, \vartheta$ *must not satisfy* at the same time all three relations (*c*), (*d*), (*e*).

If then all the above conditions are true $f(x)$ is irreducible and the group is a subgroup of G_{240}.

If $\mu = \vartheta = \delta = \nu = 0$, then our identity reduces to $2^5\gamma^2r^5 = 1$ and hence in this case γ is a perfect fifth power of a rational quantity.

σ_1 cannot be a substitution of the Galois group of $f(x) = 0$, for it produces, if applied to

$$\tau = \rho_2\rho_4^2 \neq 0,$$

$\rho_8\rho_6^2$ which is equal to zero.

σ_2 applied to $\tau = \rho_2\rho_4^2$ produces $\rho_6\rho_2^2 = 0$.

It follows then that the Galois group of $f(x) = 0$ can only be G_{10} and hence $f(x) = 0$ is an irreducible Abelian equation.

Type VI. If $\omega, \gamma, r, \tau, \kappa, \vartheta, \delta, \mu, \nu$ *but not* $\sqrt{\gamma}$, $\sqrt[5]{\kappa}$ *are in our given rational domain, and if not all five quantities* $r, \vartheta, \delta, \mu, \nu$ *vanish, and if the identity of page* 145 *exists, then* $f(x) = 0$ *with the roots on page* 145 *is an irreducible Abelian equation with the special restriction, that, if* $r = 0$ *and* $\mu = -\vartheta\kappa/\tau^2$, *then* $\nu, \delta, \vartheta, \tau, \kappa$ *must not satisfy all the three relations* (*c*), (*d*), (*e*) *of pages* 147–148.

Illustrations:

(1) $\mu = \vartheta = \delta = \nu = 0, \quad r = 2.$

To satisfy the given identity $\gamma = \frac{1}{2^5}$, τ, κ may be chosen arbitrarily according to the given conditions. The roots are:

$$x_1, x_6 = \frac{\tau}{\sqrt[5]{\kappa^2}} + \sqrt[5]{\kappa} \pm \frac{1}{2\sqrt{2}},$$

$$x_2, x_7 = \frac{\omega^4\tau}{\sqrt[5]{\kappa^2}} + \omega^3\sqrt[5]{\kappa} \pm \frac{1}{2\sqrt{2}},$$

. . . . etc. . . . .

$$f(x) = x^{10} - \frac{5}{8}x^8 - 10\tau x^7 + 5\left(\frac{1}{2^5}-\frac{2\tau^3}{\kappa}\right)x^6 - \left(2\kappa - \frac{5}{4}\tau + 2\frac{\tau^5}{\kappa^2}\right)x^5$$

$$+\left(25\tau^2-\frac{25}{8}\right)x^4+\left(\frac{125}{2^5}\tau+\frac{50\tau^4}{\kappa}-\frac{5}{2}\frac{\tau^5}{\kappa^2}-\frac{5}{2}\kappa\right)x^3$$

$$+\left(\frac{25}{2^4}\frac{\tau^3}{\kappa}-\frac{75}{2^{12}}+10\tau\cdot\kappa+35\frac{\tau^6}{\kappa^2}-\frac{75}{2^2}\tau^2\right)x^2$$

$$+\left(\frac{75}{2^8}\tau^2-\frac{75}{2^4}\frac{\tau^4}{\kappa}-\frac{5}{2^5}\kappa-\frac{5}{2^5}\frac{\tau^5}{\kappa_2}+10\tau^3+10\frac{\tau^8}{\kappa^3}\right)x$$

$$+\frac{\tau^{10}}{\kappa^4}+\kappa^2+\frac{25}{2^6}\tau^2+2\frac{\tau^5}{\kappa}-\frac{5}{2^2}\kappa\tau-2^5\cdot 5^2+\frac{25}{2^8}\frac{\tau^3}{\kappa}-\frac{35}{2^8}\frac{\tau^6}{\kappa^2}=0.$$

Assuming $\tau = 1$, $\kappa = 2$ then

$$(x) = x^{10}-\frac{5}{8}x^8-10x^7-\frac{155}{32}x^6-\frac{13}{4}x^5-\frac{175}{8}x^4+\frac{725}{2^5}x^3$$
$$+\frac{44085}{2^{12}}x^2+\frac{465}{2^8}x+\frac{405457}{2^9}=0.$$

The roots are:

$$x_1,\, x_6 = \frac{1}{\sqrt[5]{4}}+\sqrt[5]{2}\pm\frac{1}{2\sqrt{2}},$$

$$x_2,\, x_7 = \frac{\omega^4}{\sqrt[5]{4}}+\omega^3\sqrt[5]{2}\pm\frac{1}{2\sqrt{2}},$$

$$\cdot\quad\cdot\quad\cdot\quad \text{etc.}\quad\cdot\quad\cdot\quad\cdot$$

The given domain must include $R(\omega)$ but not $\sqrt{2}$.

(2) $r = \mu = \vartheta = \delta = 0$, $\nu = 1$, $\gamma = 2$.

As this is the case,

$$r = 0 \quad\text{and}\quad \mu = -\frac{\vartheta\kappa}{\tau^2} = 0,$$

we first must see that not all three relations (c), (d), (e) on pages 147–148 are satisfied. In our case (c) reduces to $\nu = 0$ and hence the values chosen are admissible. Our identity reduces to

$$\frac{1}{2^7} = \frac{\nu^5\tau^5}{\kappa^2} = \frac{\tau^5}{\kappa^2}.$$

Choosing $\kappa = 2$, then $\sqrt[5]{2}$ must not be in our domain and $\sqrt{2}$ must not in the given domain, but ω must be rationally known. We have then $\tau = \frac{1}{2}$.

$$f(x) = x^{10}+5x^7-\frac{35}{8}x^6+\frac{29}{2^6}x^5-\frac{25}{4}x^4+\frac{1575}{2^4}x^3+\frac{1727}{2^8}x^2-\frac{2075}{2^{10}}x+\frac{1165}{2^{14}}=0.$$

The roots are:

$$x_1,\, x_6 = \frac{1}{2^5\sqrt[5]{4}}+\sqrt[5]{2}\pm\frac{\sqrt{2}}{2^5\sqrt[5]{4}},$$

$$x_2,\, x_7 = \frac{\omega^4}{2^5\sqrt[5]{4}}+\omega^3\sqrt[5]{2}\pm\frac{\omega^4\sqrt{2}}{2^5\sqrt[5]{4}},$$

$$\cdot\quad\cdot\quad\cdot\quad \text{etc.}\quad\cdot\quad\cdot\quad\cdot$$

B. I. (2) $R_2 = 0$, $S_2 \neq 0$.

There is no irreducible Abelian equation of this proposed type. The proof is here omitted as it is entirely identical to the proof given under A. I. 2; we need

only to replace

$$\rho_1, \quad \rho_2, \quad \rho_3, \quad \rho_4, \quad x_i,$$

by, respectively,

$$\rho_6, \quad \rho_8, \quad \rho_2, \quad \rho_4, \quad \varphi_i.$$

B. II. $R_2 \neq 0$.

From the relation

$$\rho_6\rho_4 + \rho_2\rho_8 = 2R_2$$

we can have at most two of the four ρ's equal to zero: either $\rho_6 = \rho_4 = 0$ or $\rho_2 = \rho_8 = 0$. Both cases are identical in the abstract. We therefore cover all cases corresponding to B. II. by dividing B. II. into the three sub-cases

(1) $$\rho_8 = \rho_2 = 0,$$
(2) $$\rho_8 = 0; \quad \rho_2 \neq 0,$$
(3) $$\rho_8 \neq 0; \quad \rho_2 \neq 0.$$

B. II. (1) $\rho_8 = \rho_2 = 0, \quad \rho_4 \neq 0, \quad \rho_6 \neq 0.$

From the conditions:

$$\begin{aligned} \rho_2 = 0 \text{ follows } K_2 - L_2\sqrt{5} &= -N_2\sqrt{-10+2\sqrt{5}} + M_2\sqrt{-10-2\sqrt{5}}, \\ \rho_8 = 0 \text{ follows } K_2 - L_2\sqrt{5} &= +N_2\sqrt{-10+2\sqrt{5}} - M_2\sqrt{-10-2\sqrt{5}}, \\ \hline \therefore K_2 - L_2\sqrt{5} = 0 \quad &\text{or} \quad K_2 = L_2\sqrt{5}. \end{aligned}$$

Suppose for a moment $K_2 = L_2 = 0$; then $\rho_6 = -\rho_4$ and $\varphi_1 = x_1 + x_6 = 0$, which is impossible; hence $K = L\sqrt{5} \neq 0$ and therefore $\sqrt{5}$ *is in our given domain.*

It further follows from the above, that

$$\frac{N_2}{M_2} = \frac{\sqrt{-10-2\sqrt{5}}}{\sqrt{-10+2\sqrt{5}}},$$

or

$$N_2 = -M_2\frac{1+\sqrt{5}}{2},$$

so that

$$\rho_6^5 = 2K_2 - \frac{20M_2}{\sqrt{-10+2\sqrt{5}}}$$

and

$$\rho_4^5 = 2K_2 + \frac{20M_2}{\sqrt{-10+2\sqrt{5}}};$$

hence

$$\rho_4\rho_6 = 2R_2 = \sqrt[5]{4K_2^2 - \frac{400}{-10+2\sqrt{5}}M_2^2},$$

or, putting

$$M_2' = \frac{20}{-10+2\sqrt{5}}M_2,$$

then M_2' *is rationally known* and

$$(2R_2)^5 = 4K_2^2 - M_2'(-10+2\sqrt{5}) \textit{ is a perfect fifth power.}$$

The following four functions ψ_1/ψ_6, ψ_3/ψ_4^2, ψ_7/ψ_6^2, ψ_9/ψ_4 are seen to be double-valued

functions under G_{10}; hence, if we write

$$\rho_1 = \pi\rho_6, \quad \rho_3 = \vartheta\rho_4^2, \quad \rho_7 = \sigma\rho_6^2, \quad \rho_9 = \delta\rho_4,$$

then π, ϑ, σ, δ are quantities in our given rational domain after the adjunction of ω.

By actual computation we have

$$\psi_1\psi_9 = 10^2\gamma\rho_1\rho_9 = i\sum_1^5 \varphi^2_{i+5} + (\omega + \omega^4)F_1 + (\omega^2 + \omega^3)F_2,$$

$$\psi_3\psi_7 = 10^2\gamma\rho_3\rho_7 = i\sum_1^5 \varphi^2_{i+5} + (\omega^2 + \omega^3)F_1 + (\omega + \omega^4)F_2,$$

where

$$F_1 = \varphi_6\varphi_8 + \varphi_6\varphi_9 + \varphi_7\varphi_9 + \varphi_7\varphi_{10} + \varphi_8\varphi_{10},$$
$$F_2 = \varphi_6\varphi_7 + \varphi_6\varphi_{10} + \varphi_7\varphi_8 + \varphi_8\varphi_9 + \varphi_9\varphi_{10}.$$

F_1, F_2 are seen to be rationally known, if applied to by G_{10} and as $\sqrt{5}$ is in our given domain, the two functions $\rho_1\rho_9$ and $\rho_3\rho_7$ are rationally known and hence from

$$\rho_1\rho_9 = \pi\delta\rho_4\rho_6 = 2\pi\delta R_2$$

and

$$\rho_3\rho_7 = \sigma\vartheta\rho_4^2\rho_6^2 = 4\sigma\vartheta R_2^2$$

it follows that $\pi\delta$ and $\sigma\vartheta$ are in our given domain.

Further,

$$\rho_1^5 + \rho_9^5 = (K_1 + L_1\sqrt{5})2 = \delta^5\rho_4^5 + \rho_6^5\pi^5$$
$$= 2K_2(\pi^5 + \delta^5) + M_2'\sqrt{-10 + 2\sqrt{5}}(\delta^5 - \pi^5).$$

Similarly

$$\rho_3^5 + \rho_7^5 = (K_1 - L_1\sqrt{5})2 = \vartheta^5\rho_4^{10} + \sigma^5\rho_6^{10}$$
$$= [4K_2^2 + M_2'^2(-10 + 2\sqrt{5})](\vartheta^5 + \sigma^5) + 2^2K_2M_2'\sqrt{-10 + 2\sqrt{5}}(\vartheta^5 - \sigma^5).$$

Summarizing now the facts that $\rho_1^5 + \rho_9^5$, $\rho_3^5 + \rho_7^5$, $\sigma\vartheta$, $\delta\pi$ are quantities in our given domain and that δ, π, σ, ϑ are also in our domain, if we adjoin ω to this domain, then it is seen that we may write

(1) $\pi = a + b\sqrt{-10 + 2\sqrt{5}}$, (2) $\delta = a - b\sqrt{-10 + 2\sqrt{5}}$,

(3) $\vartheta = \alpha + \beta\sqrt{-10 + 2\sqrt{5}}$, (4) $\sigma = \alpha - \beta\sqrt{-10 + 2\sqrt{5}}$,

where a, b, α, β *are in our given rational domain.* The relation

$$(2R_2)^5 = 4K_2^2 - M_2'^2(-10 + 2\sqrt{5}) \tag{5}$$

which we established above and the identity

$$\begin{aligned}\frac{1}{2^5\gamma^2} = {} & \pi^5\rho_6^5 + \delta^5\rho_4^5 + \vartheta^5\rho_4^{10} + \sigma^5\rho_6^{10} + r^5 + 5\cdot 2R_2(\vartheta\sigma 2R_2 - \pi\delta)[(\pi^2\vartheta \\ & + \delta^2\sigma)4R_2^4 - (\vartheta^2\delta\rho_4^5 + \sigma^2\pi\rho_6^5)] + 5r\cdot 4R_2^2(\pi^2\delta^2 + \vartheta^2\sigma^2) \\ & + 5r^2[(\pi^2\vartheta + \delta^2\sigma)4R_2^2 + (\vartheta^2\delta\rho_4^5 + \sigma^2\pi\rho_6^5)] - 5r\cdot 2R_2(\pi\delta + \vartheta\sigma\cdot 2R_2) \\ & - 5r\pi\delta\vartheta\sigma\cdot 8R_2^3 - 5r(\pi^3\sigma\rho_6^5 + \delta^3\vartheta\rho_4^5) - 5r\cdot 2R_2(\vartheta^3\pi\rho_4^5 + \sigma^3\delta\rho_6^5),\end{aligned} \tag{6}$$

which is based upon the condition

$$\prod_{i=1}^{i=5}(x_i - x_{i+5}) = \sqrt{\gamma},$$

enable us to express M_2' and K_2 in terms of $R_2, \sigma, \vartheta, \pi, \delta, \gamma$. The relation (6) does not involve any quantity lying outside of our rational domain, for we have

$$\pi\delta = a^2 - b^2(-10 + 2\sqrt{5}),$$

$$\vartheta\sigma = \alpha^2 - \beta^2(-10 + 2\sqrt{5}),$$

$$\pi^2\vartheta + \delta^2\sigma = 2\alpha[a^2 + b^2(-10 + 2\sqrt{5})] + 4\beta ab(-10 + 2\sqrt{5}),$$

$$\begin{aligned}\vartheta^2\delta\rho_4{}^5 + \sigma^2\pi\rho_6{}^5 = {} & 4K_2[a\alpha^2 + a\beta^2(-10 + 2\sqrt{5}) - 2\alpha\beta b(-10 + 2\sqrt{5})] \\ & + 2M_2'[2a\alpha\beta - b\alpha^2 - b\beta^2(-10 + 2\sqrt{5})](-10 + 2\sqrt{5}),\end{aligned}$$

$$\begin{aligned}\pi^3\sigma\rho_6{}^5 + \delta^3\vartheta\rho_4{}^5 = {} & 4K_2\alpha a^3 + (-10 + 2\sqrt{5})2[\sigma ab\alpha K_2 \\ & - (3a^2b + b^3[-10 + 2\sqrt{5}])(2\beta K_2 + M_2') \\ & + \beta M_2'(a^3 + 3ab[-10 + 2\sqrt{5}])],\end{aligned}$$

$$\begin{aligned}\vartheta^3\pi\rho_4{}^5 + \sigma^3\delta\rho_6{}^5 = {} & 4K_2a\alpha + 2(-10 + 2\sqrt{5})[\sigma a\alpha\beta^2K_2 + 6b\alpha^2\beta K_2 \\ & + 2K_2b\beta^2(-10 + 2\sqrt{5}) + M_2'(b\alpha - 3b\alpha\beta^2[-10 + 2\sqrt{5}] \\ & + 3a\alpha^2\beta + a\beta^2(-10 + 2\sqrt{5})],\end{aligned}$$

$$\begin{aligned}\pi^5\rho_6{}^5 + \delta^5\rho_4{}^5 = {} & 4K_2[a^5 + 10a^3b[-10 + 2\sqrt{5}] + 5ab^4[-10 + 2\sqrt{5}]^2] \\ & - 2M_2'(-10 + 2\sqrt{5})[5a^4b + 10a^2b^3[-10 + 2\sqrt{5}] \\ & + b^5[-10 + 2\sqrt{5}]^2],\end{aligned}$$

$$\begin{aligned}\vartheta^5\rho_4{}^{10} + \sigma^5\rho_6{}^{10} = {} & 2[4K_2{}^2 + M_2'^2(-10 + 2\sqrt{5})][\alpha^5 + \alpha^3\beta^2(-10 + 2\sqrt{5}) \\ & + \alpha\beta^4(-10 + 2\sqrt{5})^2] + 8K_2 + M_2'[\alpha^4\beta + \alpha^2\beta^3(-10 + 2\sqrt{5}) \\ & + \beta^5(-10 + 2\sqrt{5})^2](-10 + 2\sqrt{5}).\end{aligned}$$

Having found K_2 and M_2' from (5) and (6), we know ρ_4 and ρ_6.

The roots of $f(x) = 0$ are

$$\begin{aligned}x_1, x_6 = {} & \rho_4 + \rho_6 \pm \sqrt{\gamma}[(a + b\sqrt{-10 + 2\sqrt{5}})\rho_6 + (\alpha + \beta\sqrt{-10 + 2\sqrt{5}})\rho_4{}^2 \\ & + (\alpha - \beta\sqrt{-10 + 2\sqrt{5}})\rho_6{}^2 + (a - b\sqrt{-10 + 2\sqrt{5}})\rho_4 + r],\end{aligned}$$

$$\begin{aligned}x_2, x_7 = {} & \omega^2\rho_6 + \omega^3\rho_4 \pm \sqrt{\gamma}[\omega^2(a + b\sqrt{-10 + 2\sqrt{5}})\rho_6 + \omega(\alpha + \beta\sqrt{-10 + 2\sqrt{5}})\rho_4{}^2 \\ & + (\alpha - \beta\sqrt{-10 + 2\sqrt{5}})\omega^4\rho_6{}^2 + \omega^3(a - b\sqrt{-10 + 2\sqrt{5}})\rho_4 + r],\end{aligned}$$

$$\cdot \quad \cdot \quad \cdot \quad \text{etc.,} \quad \cdot \quad \cdot \quad \cdot$$

where

$$\rho_4 = 2K_2 + M_2'\sqrt{-10 + 2\sqrt{5}},$$

$$\rho_6 = 2K_2 - M_2'\sqrt{-10 + 2\sqrt{5}},$$

where K_2 and M_2' are found from the relations (5) and (6).

The corresponding equation is

$$f(x) = (x^5 - A_1x^4 + A_2x^3 - A_3x^3 + A_4x - A_5) \times (x^5 - B_1x^4 + B_2x^3 - B_3x^2 + B_4x - B_5) = 0,$$

where

$$A_1, B_1 = \pm 5r\sqrt{\gamma},$$

$$A_2, B_2 = 10\gamma r^2 - 10R_2 - 10R_2\gamma(\pi\delta + 2\sigma\vartheta R_2) \mp 10R_2\sqrt{\gamma}(\pi + \delta),$$

$$\begin{aligned} A_3, B_3 = {} & 5\gamma(\sigma^2\rho_6^5 + \vartheta^2\rho_4^5) - 15r\gamma \cdot 2R_2(\pi\delta + \vartheta\sigma) + 10\gamma(\pi\vartheta + \sigma\delta)4R_2^2 \\ & \pm \sqrt{\gamma}5[2\gamma r^3 + 4R_2^2(\sigma + \vartheta) + \gamma(\sigma^2\pi\rho_6^5 + \vartheta^2\delta\rho_4^5) + 4\gamma R_2^2(\delta^2\sigma + \pi^2\vartheta) \\ & - 6rR_2 - 6rR_2(\pi\delta + \sigma\vartheta 2R_2)], \end{aligned}$$

$$\begin{aligned} A_4, B_4 = {} & + 20R_2^2 + 5\gamma^2r^4 - 30\gamma r^2R_2 + 20\gamma^2R_2^2(\pi^2\delta^2 + 4\sigma^2\vartheta^2R_2^2) \\ & - 5\gamma^2[(\sigma^3\delta\rho_6^5 + \vartheta^3\pi\rho_4^5)2R_2 + \delta^3\vartheta\rho_4^5 + \pi^3\sigma\rho_6^5)] + 20\gamma R_2^2(\pi^2 + \delta^2) \\ & - 30r^2\gamma^2R_2(\pi\delta + 2\sigma\vartheta R_2) + 10\gamma^2r[\sigma^2\pi\rho_6^5 + \vartheta^2\delta\rho_4^5 + 4R_2^2(\delta^2\sigma + \pi^2\vartheta)] \\ & + 40\gamma rR_2^2(\sigma + \vartheta) - 40\gamma\sigma\vartheta R_2^3(1 + \gamma\pi\delta) - 15\gamma(\pi\sigma\rho_6^5 + \vartheta\delta\rho_4^5) \\ & + 80\gamma\pi\delta R_2^2 \pm 5\sqrt{\gamma}[- 2\gamma R_2(\sigma^3\rho_6^5 + \vartheta^3\rho_4^5) + 2\gamma r(\sigma^2\rho_6^5 + \vartheta^2\rho_4^5) \\ & - (\vartheta\rho_4^5 + \sigma\rho_6^5) - 6r^2\gamma R_2(\pi + \delta) - 8\gamma(\pi + \delta)\vartheta\sigma R_2^3 \\ & - 3\gamma(\pi^2\sigma\rho_6^5 + \delta^2\vartheta\rho_4^5) + 8\gamma R_2^2(\pi + \delta)(1 + \pi\delta) + 32r\gamma R_2^2(\pi\vartheta + \sigma\delta)], \end{aligned}$$

$$\begin{aligned} A_5, B_5 = {} & 4K_2 + 5\gamma^2(\pi^4\rho_6^5 + \delta^4\rho_4^5) + 10\gamma(\pi^2\rho_6^5 + \delta^2\rho_4^5) \\ & - 5\gamma^2[2R_2(\sigma\delta^3 + \pi^3\vartheta) + \sigma\vartheta(\vartheta^2\rho_4^5 + \sigma^2\rho_6^5)]4R_2^2 \\ & - 40\gamma R_2^3(\sigma\pi + \delta\vartheta) - 120\gamma R_2^3(\sigma\delta + \pi\vartheta) - 120\gamma^2R_2^3\pi\delta(\sigma\delta + \vartheta\pi) \\ & + 10\gamma R_2(\sigma^2\rho_6^5 + \vartheta^2\rho_4^5) + 10\gamma^2R_2(\sigma^2\pi^2\rho_6^5 + \delta^2\vartheta^2\rho_4^5) \\ & - 10\gamma^2rR_2(\pi + \delta) + 5\gamma^2r^2(\sigma^2\rho_6^5 + \vartheta^2\rho_4^5) + 40r^2\gamma^2R_2^2(\delta\sigma + \pi\vartheta) \\ & - 10r\gamma^2R_2(\sigma^3\rho_6^5 + \vartheta^3\rho_4^5) - 5r\gamma(\vartheta\rho_4^5 + \sigma\rho_6^5) - 40r\gamma^2\vartheta\sigma R_2^3(\pi + \delta) \\ & - 15r\gamma(\vartheta\delta\rho_4^5 + \sigma\pi\rho_6^5) - 15r\gamma^2(\vartheta\delta^2\rho_4^5 + \sigma\pi^2\rho_6^5) \\ & + 40r\gamma R_2^2(\pi + \delta) + 40r\gamma^2R_2^2\pi\delta(\pi + \delta) \\ & \pm \sqrt{\gamma}[\gamma^2(\pi^5\rho_6^5 + \vartheta^5\rho_4^{10} + \sigma^5\rho_6^{10} + \delta^5\rho_4^5 + r^5) + 5(\pi\rho_6^5 + \delta\rho_4^5) \\ & + 10\gamma(\pi^3\rho_6^5 + \delta^3\rho_4^5) - 20\gamma^2R_2^2[2R_2\pi\delta(\sigma\delta^2 + \vartheta\pi^2) + \vartheta\sigma(\delta\vartheta^2\rho_4^5 + \sigma^2\pi\rho_6^5)] \\ & - 40R_2^3(\sigma + \vartheta) - 120\gamma R_2^3\pi\delta(\sigma + \vartheta) - 120\gamma R_2^3(\sigma\delta^2 + \vartheta\pi^2) \\ & + 10\gamma^2R_2(\pi\delta(\sigma^2\pi\rho_6^5 + \vartheta^2\delta\rho_4^5) + 8R_2^3\sigma\vartheta(\sigma\delta^2 + \vartheta\pi^2)) \\ & + 10\gamma R_2(\sigma\vartheta 8R_2^3(\sigma + \vartheta) + \sigma^2\delta\rho_6^5 + \pi\vartheta^2\rho_4^5) - 10\gamma r^3R_2 \\ & - 10\gamma^2r^3R_2(\pi\delta + 2R_2\sigma\vartheta) + 5\gamma^2(\sigma^2\pi\rho_6^5 + \vartheta^2\delta\rho_4^5)r^2 \\ & + 20\gamma^2r^2R_2^2(\delta^2\sigma + \pi^2\vartheta) + 20\gamma r^2R_2^2(\sigma + \vartheta) + 20rR_2^2 \\ & - 5r\gamma^2(\delta^3\vartheta\rho_4^5 + \pi^3\sigma\rho_6^5) - 10r\gamma^2R_2(\sigma^3\delta\rho_6^5 + \vartheta^3\pi\rho_4^5) \\ & + 20\pi\gamma^2R_2^2(\pi^2\delta^2 + 4R_2^2\sigma^2\vartheta^2) - 40r\gamma^2R_2^3\sigma\vartheta\pi\delta \\ & + 20\gamma R_2^2(\pi^2 + \delta^2) - 40r\gamma R_2^3\sigma\vartheta + 80r\gamma R_2^2\pi\delta]. \end{aligned}$$

Our next question is: Are the derived conditions also sufficient, that $f(x) = 0$ with the above roots is an irreducible Abelian equation?

First of all we note that *not all five quantities a, b, α, β and r can be zero at the* same time, for then $f(x)$ would contain the two rational factors

$$x^5 - A_1x^4 + A_2x^3 - A_3x^2 + A_4x - A_5$$

and

$$x^5 - B_1x^4 + B_2x^3 - B_3x^2 + B_4x - B_5,$$

as seen from the above functions A_i, B_i. Further

$$\rho_4 \neq \pm \rho_6,$$

for then

$$\rho_4\rho_6 = \pm \rho_4^2 = \pm \rho_6^2 = 2R_2$$

and hence

$$\varphi_1 = x_1 + x_6$$

would be rationally known. It follows then that equations (5) and (6) must lead to

$$K_2 \neq 0 \quad \text{and} \quad M_2' \neq 0$$

otherwise we have no irreducible equation corresponding to case B. ω may be rationally known or not; it certainly cannot be expressed rationally by ρ_4 or ρ_6, for if ω is not in our domain, then ω satisfies an irreducible quadratic, as $\sqrt{5}$ is rational, whereas ρ_4 satisfies an irreducible quintic. It follows then, as just explained under B. I, that the sum of any two, three, or four roots must always contain the irrationality ρ_4 and ρ_6 respectively and that $f(x)$ cannot have a rational factor of the first, second, third, and fourth degree.

Whenever the sum of any five roots does not vanish, this sum is irrational for the same reason; the first case, however, can happen only if we pick out five roots belonging to the same system of intransitivity. Our question of the irreducibility of $f(x)$ simply amounts to the following question: Under what conditions can all five A_i be rationally known?

A_1 shows that we must have $r = 0$, if $\sqrt{\gamma}$ shall disappear. Further, from A_2, we must have $\pi + \delta = 0$ as $R_2 \neq 0$; hence

$$a = 0.$$

Under these conditions we get further the relations from A_3

$$\text{(I)} \qquad \frac{4\alpha R_2^2}{-10 + 2\sqrt{5}} = \frac{4K_2b\alpha\beta + M_2'b(\alpha^2 + \beta^2[-10 + 2\sqrt{5}])}{1 + b^2(-10 + 2\sqrt{5})},$$

from A_4:

$$\text{(II)} \qquad \begin{aligned} &4\gamma R_2K_2[\alpha^3 + 3\alpha\beta^2(-10 + 2\sqrt{5})] + 2\gamma R_2M_2'(-10 + 2\sqrt{5})[3\alpha^2\beta \\ &\quad + \beta^3(-10 + 2\sqrt{5})] + [2K_2\alpha + \beta M_2'(-10 + 2\sqrt{5})] \\ &\qquad \times [1 + 3\gamma b^2(-10 + 2\sqrt{5})] = 0, \end{aligned}$$

and from A_5:

$$\text{(III)} \qquad \begin{aligned} &\gamma^2\{2b^5M_2'(-10 + 2\sqrt{5})^3 + [8K_2^2 + 2M_2'^2(-10 + 2\sqrt{5})] \\ &\times[\alpha^5 + 10\alpha^3\beta^2(-10 + 2\sqrt{5}) + 5\alpha\beta^4(-10 + 2\sqrt{5})^2] \\ &+ 8K_2M_2'(-10 + 2\sqrt{5})[5\alpha^4\beta + 10\alpha^2\beta^2(-10 + 2\sqrt{5}) \\ &+ \beta^5(-10 + 2\sqrt{5})\} - 10M_2'(-10 + 2\sqrt{5})(b + 2b^3\gamma) \\ &- 80R_2^3\alpha[\gamma^2b^4(-10 + 2\sqrt{5}) - 1] + 10R_2b(-10 + 2\sqrt{5})[8K_2^2\alpha\beta \\ &+ 2M_2'\alpha^2 + 2M_2'\beta^2(-10 + 2\sqrt{5})][2\gamma^2R_2(\alpha^2 - \beta^2[-10 + 2\sqrt{5}]) \\ &- b^2(-10 + 2\sqrt{5}) - \gamma] + 160R_2^4\gamma\alpha[\alpha^2 - \beta^2(-10 + 2\sqrt{5})] \\ &\times[1 + \gamma b^2(-10 + 2\sqrt{5})] = 0. \end{aligned}$$

If then $r = a = 0$ and equations I, II, III are true $f(x)$ is reducible, having two rational factors each of the fifth degree. If, however, not all these five equations are satisfied, $f(x)$ is irreducible and must have two as well as five systems of intransitivity, as seen directly from the A_i, B_i respectively from the roots. Its group, being therefore a subgroup of G_{240}, can only be G_{10}, as σ_1 and σ_2 are impossible in the Galois group of $f(x) = 0$ under our conditions, for we have

σ_1 interchanges ρ_4 and ρ_6, so that $\rho_4 = \pm \rho_6$, if σ_1 were in our Galois group and hence $M_2' = 0$ or $K_2 = 0$, but this contradicts our conditions.

Similarly σ_2 is impossible in the Galois group, as σ_2 changes ρ_4 to $\omega^4\rho_2 = 0$, which contradicts $R_2 \neq 0$. It follows then that our $f(x) = 0$ is an irreducible Abelian equation under the derived conditions.

Type VII. If $\sqrt{5}$, α, β, a, b, K_2, M_2', γ, r, R_2, *but not* $\sqrt{\gamma}$ *are quantities in our given rational domain and if not all five quantities* α, β, a, b, r *are zero and if relations* (5) *and* (6) *on page* 151 *exist and if* $R_2 \neq 0$, $K_2 \neq 0$, $M_2' \neq 0$—*then the above* $f(x) = 0$ *with the given roots is an irreducible Abelian equation, if*

$$\sqrt[5]{2K_2 \pm M_2'\sqrt{-10 + 2\sqrt{5}}}$$

is not in our domain and if not all five equations $r = a = 0$, (I), (II), (III), *are satisfied.*

Illustrations:

(1) $$a = b = \alpha = \beta = 0, \qquad r = 2, \qquad M_2' = 1.$$

Our identity (6) shows

$$\frac{1}{2^5\gamma^2} = r^5;$$

hence $\gamma = 2^{-5}$.

Equation (5) shows

$$(2R_2)^5 = 4K_2^2 - (-10 + 2\sqrt{5}),$$

hence, taking

$$K_2 = +\sqrt{\frac{\sqrt{5}}{2}},$$

we have

$$R_2 = \tfrac{1}{2}\sqrt[5]{10}.$$

Here our rational domain cannot include $\sqrt{2}$, and

$$\sqrt[5]{2\sqrt{\frac{\sqrt{5}}{2}} \pm \sqrt{-10 + 2\sqrt{5}}},$$

but must contain $\sqrt{5}$, $\sqrt[5]{10}$, $\sqrt{\dfrac{\sqrt{5}}{2}}$.

As $r \neq 0$, not all five equations $r = \alpha = 0$, (I), (II), (III) are satisfied and hence all conditions are satisfied. The roots are

$$x_1, x_6 = \sqrt[5]{\sqrt{2\sqrt{5}} + \sqrt{-10 + 2\sqrt{5}}} + \sqrt[5]{\sqrt{2\sqrt{5}} - \sqrt{-10 + 2\sqrt{5}}} \pm \sqrt{\frac{1}{2^3}},$$

$$x_2, x_7 = \omega^3 \sqrt[5]{\sqrt{2\sqrt{5}} + \sqrt{-10 + 2\sqrt{5}}} + \omega^2 \sqrt[5]{\sqrt{2\sqrt{5}} - \sqrt{-10 + 2\sqrt{5}}} \pm \sqrt{\frac{1}{2^3}}$$

. . . . etc. . . .

$$f(x) = x^{10} - \left(\frac{1}{8} + 2\sqrt[5]{10}\right)5x^8 + \left(\frac{9}{8} - 16\sqrt[5]{10} + 25\sqrt[5]{10^2}\right)x^6 - 4\sqrt{2\sqrt{5}}x^5$$

$$+ \frac{1}{8}\left(\frac{1}{20} - 308\sqrt[5]{10^2} + \frac{119}{2}\sqrt[5]{10}\right)x^4 + 5(1 - 4\sqrt[5]{10})\sqrt{2\sqrt{5}}x^3$$

$$+\left(\frac{189}{5\cdot 2^{12}}-\frac{1}{10}\sqrt[5]{10}+\frac{45}{16}\sqrt[5]{10^2}-\frac{743}{2^7}\sqrt[5]{10^3}\right)x^2-5\sqrt{2\sqrt{5}}\left(\sqrt[5]{10}+\frac{1}{16}\right)x$$

$$+8\sqrt{5}+\frac{1}{5^2\cdot 2^{12}}-\frac{1}{5\cdot 2^8}\sqrt[5]{10}+\frac{7}{5\cdot 2^6}\sqrt[5]{10^2}+\frac{1}{4}\sqrt[5]{10^3}+\sqrt[5]{10^4}=0.$$

(2) $$r=\alpha=\beta=b=0,\qquad a=1,\qquad M_2'=1,$$

$$R_2=-\frac{1}{2}\sqrt[5]{2\sqrt{5}},\quad K_2=\frac{1}{2}\sqrt{-10},\quad \gamma=\frac{1}{2^3\sqrt[4]{-10}}.$$

By these values chosen all conditions are satisfied.

The roots are:

$$x_1, x_6=\sqrt[5]{\sqrt{-10}+\sqrt{-10+2\sqrt{5}}}+\sqrt[5]{\sqrt{-10}-\sqrt{-10+2\sqrt{5}}}$$

$$\pm\frac{1}{2}\sqrt{\frac{1}{2\sqrt[4]{-10}}}\left[\sqrt[5]{\sqrt{-10}+\sqrt{-10+2\sqrt{5}}}+\sqrt[5]{\sqrt{-10}-\sqrt{-10+2\sqrt{5}}}\right],$$

$$x_2, x_7=\omega^3\sqrt[5]{\sqrt{-10}+\sqrt{-10+2\sqrt{5}}}+\omega^2\sqrt[5]{\sqrt{-10}-\sqrt{-10+2\sqrt{5}}}$$

$$\pm\frac{1}{2}\sqrt{\frac{1}{2\sqrt[4]{-10}}}\left[\omega^3\sqrt[5]{\sqrt{-10}+\sqrt{-10+2\sqrt{5}}}+\omega^2\sqrt[5]{\sqrt{-10}-\sqrt{-10+2\sqrt{5}}}\right],$$

. . . etc. . . .

$$f(x)=x^{10}+2\sqrt[5]{2\sqrt{5}}\left(5+\frac{1}{8\sqrt[4]{-10}}\right)x^8+\sqrt[5]{20}\left[45+\frac{1}{2^6\sqrt{-10}}-\frac{30}{2^3\sqrt[4]{-10}}\right]x^6$$

$$+4\sqrt{-10}\left[1+\frac{5}{4\sqrt[4]{-10}}+\frac{5}{2^6\sqrt{-10}}\right]x^5+5\sqrt[4]{40\sqrt{5}}$$

$$\times\left[20+\frac{9}{\sqrt[4]{-10}}+\frac{55}{2^6\sqrt{-10}}\right]x^4+40\sqrt{-10}\sqrt[5]{2\sqrt{5}}\left[1+\frac{21}{8\sqrt[4]{-10}}\right.$$

$$\left.+\frac{27}{2^6\sqrt{-10}}+\frac{3}{2^9\sqrt[4]{-10^3}}\right]x^3+100\sqrt[5]{2^4\cdot 5^2}\left[1+\frac{5}{2^3\sqrt[4]{-10}}+\frac{7}{2^6\sqrt{-10}}\right.$$

$$\left.-\frac{1}{2^9\sqrt[4]{-10^3}}\right]x^2+40\sqrt{-10}\sqrt[5]{20}\left[1+\frac{1}{4\sqrt{-10}}+\frac{5}{2^4\sqrt{-10}}\right.$$

$$\left.+\frac{1}{2^7\sqrt[4]{-10^3}}+\frac{1}{5\cdot 2^{13}}\right]x+8\sqrt{-10}\left[26+\frac{15}{2\sqrt[4]{-10}}+\frac{55}{2^4\sqrt{-10}}\right.$$

$$\left.+\frac{5}{2^7\sqrt[4]{-10^3}}-\frac{13}{2^{12}\cdot 5}\right]=0.$$

Here the given rational domain must not include

$$\sqrt{2\sqrt[4]{-10}},\quad \sqrt[5]{\sqrt{-10}\pm\sqrt{-10+2\sqrt{5}}},$$

but it must contain $\sqrt[4]{-10}$, $\sqrt[5]{2\sqrt{5}}$.

B. II. (2) $\rho_8=0,\quad \rho_2\neq 0,\quad \rho_4\neq 0,\quad \rho_6\neq 0.$

From $\rho_8=0$ follows

$$K_2-L_2\sqrt{5}=+N_2\sqrt{-10+2\sqrt{5}}-M_2\sqrt{-10-2\sqrt{5}},$$

hence

$$\rho_2 = \sqrt[5]{2(K_2 - L_2\sqrt{5})} \neq 0.$$

It follows then ω *is rationally known.*

Putting then

$$\frac{\rho_1}{\rho_6} = \pi, \quad \frac{\rho_3}{\rho_4^2} = \zeta, \quad \frac{\rho_9}{\rho_4} = \delta, \quad \frac{\rho_7}{\rho_2} = \sigma,$$

then π, ζ, δ, σ *are in the given domain.*

Here again *not all five quantities* r, π, ζ, δ, σ *can be zero at once,* for then $f(x)$ would be reducible. As ω is rationally known, we may put $\rho_2^2/\rho_4 = g$ and $\rho_6^2/\rho_2 = h$, where g, h *are quantities in our given domain,* $g \neq 0$, $h \neq 0$.

We have then

$$\rho_6^5 = \frac{\rho_2^2}{\rho_4}\left(\frac{\rho_6^2}{\rho_2}\right)^2 \rho_6\rho_4 = +\, 2gh^2R_2,$$

$$\rho_2^5 = \left(\frac{\rho_2^2}{\rho_4}\right)^2 \frac{\rho_2}{\rho_6^2}\cdot(\rho_4\rho_6)^2 = +\,\frac{4g^2R_2^2}{h},$$

$$\rho_4^5 = \frac{(\rho_6\rho_4)^5}{\rho_6^5} = +\,\frac{2^4R_2^4}{gh^2}.$$

If ρ_6 were rationally known, all three ρ_i would be rational and if ρ_6 is not in our given rational domain, then ρ_2 and ρ_4 cannot be rationally known. Whenever ρ_i is in our given rational domain, $f(x)$ splits up into the two rational factors

$$\prod_{i=1}^{i=5}(x - x_i) \quad \text{and} \quad \prod_{i=1}^{i=5}(x - x_{i+5}).$$

It follows then that $2gh^2R_2$ *is not a perfect fifth power.*

The last condition includes the three conditions above, namely

$$R_2 \neq 0, \quad h \neq 0, \quad g \neq 0.$$

We have here again an *identity,* based upon

$$\prod_{i=1}^{i=5}(x_i - x_{i+5}) = \sqrt{\gamma}$$

namely

$$\frac{1}{2^5\gamma^2} = 2\pi^5gh^2R_2 + \frac{4\sigma^5g^2R_2^4}{h} + \frac{2^8\zeta^5R_2^8}{g^2h^4} + \frac{2^4\delta^5R_2^4}{gh^2}$$
$$- 40R_2^3\left(\pi^3\zeta\delta + \frac{8\zeta^3\sigma\delta R_2^3}{h^3g} + \frac{\sigma^3\pi\zeta g}{h} + \frac{\delta^3\pi\sigma}{h}\right)$$
$$+ 20R_2^2\left(\frac{2^3\pi\zeta^2\delta^2R_2^3}{gh^2} + \frac{4\zeta\sigma^2\delta^2R_2^2}{h^2} + \frac{2^2\sigma\pi^2\zeta^2R_2^2}{h} + \delta\pi^2\sigma^2g\right)$$
$$+ 20rR_2^2\left(\pi^2\delta^2 + \zeta^2\sigma^2\frac{4R_2^2}{h^2}\right) + 10rR_2\left(2\pi^2\zeta R_2 + \frac{8\zeta^2\delta R_2^3}{gh^2} + \sigma^2\pi g\right.$$
$$\left. + \frac{2\delta^2\sigma R_2}{h}\right) - 10r^3R_2\left(\pi\delta + \frac{2\sigma\zeta R_2}{h}\right) - 10rR_2\left(\pi^3\zeta hg + \frac{2^4\zeta^3\pi R_2^4}{gh^2}\right.$$
$$\left. + \frac{2\sigma^3\delta gR_2}{h} + \frac{2^3\delta^3\zeta R_2^3}{gh^2}\right) - \frac{40r\pi\delta\zeta\sigma R_2^3}{h} + r^5.$$

The roots of $f(x) = 0$ are here

$$x_1,\, x_6 = \sqrt[5]{2gh^2R_2} + \sqrt[5]{\frac{4g^2R_2^2}{h}} + \sqrt[5]{\frac{2^4R_2^4}{gh^2}} \pm \sqrt{\gamma}\left[\pi\sqrt[5]{2gh^2R_2} + \zeta\sqrt[5]{\frac{2^8R_2^8}{g^2h^4}} + \sigma\sqrt[5]{\frac{4g^2R_2^2}{h}} + \delta\sqrt[5]{\frac{2^4R_2^4}{gh^2}} + r\right],$$

$$\cdot\quad\cdot\quad\cdot\quad \text{etc.}\quad\cdot\quad\cdot\quad\cdot$$

$$f(x) = (x^5 - A_1x^4 + A_2x^3 - A_3x^2 + A_4x - A_5) \times (x^5 - B_1x^4 + B_2x^3 - B_3x^2 + B_4x - B_5) = 0,$$

where

$$A_1,\, B_1 = \pm 5r\sqrt{\gamma},$$

$$A_2,\, B_2 = +10\gamma r^2 - 10R_2 - 10\gamma R_2(\pi\delta + \zeta\sigma\cdot 2R_2) \mp \sqrt{\gamma}10R_2\left(\pi + \delta + \frac{\zeta 2R_2}{h}\right),$$

$$\begin{aligned}
A_3,\, B_3 = {} & 10R_2\left(g + \frac{2R_2}{h}\right) + 10\gamma R_2\left(\sigma^2 g + \frac{\delta^2 2R_2}{h} + \frac{8R_2^3\zeta^2}{gh^2}\right) \\
& - 30\gamma r R_2\left(\pi + \frac{2R_2\zeta}{h} + \delta\right) + 20\gamma R_2\left(\pi\zeta 2R_2 + \pi\sigma g + \frac{\sigma\delta 2R_2}{h}\right) \\
& \pm \sqrt{\gamma}\left[10\gamma r^3 + 10R_2\left(\pi g + \frac{2\sigma R_2}{h} + 2\zeta R_2\right)\right. \\
& + 10\gamma R_2\left(\sigma^2\pi g + \frac{\delta^2\sigma 2R_2}{h} + \pi^2\zeta 2R_2 + \frac{8R_2^3\zeta^2\delta}{gh^2}\right) \\
& \left. - 30rR_2\left(1 + \gamma\pi\delta + \gamma\frac{\zeta\sigma 2R_2}{h}\right) + 20R_2\left(\frac{2R_2\delta}{h} + \sigma g\right)\right],
\end{aligned}$$

$$\begin{aligned}
A_4,\, B_4 = {} & -10R_2\left(\frac{2R_2 g}{h} + hg\right) + 20R_2^2 + 5\gamma^2r^4 - 30\gamma r^2R_2 \\
& + 20\gamma^2R_2^2\left(\pi^2\delta^2 + \frac{4R_2^2\zeta^2\sigma^2}{h^2}\right) - 10\gamma^2R_2\left(\frac{\sigma^3\delta 2R_2 g}{h} + \frac{8R_2^3\delta^3\zeta}{gh^2}\right. \\
& \left. + \pi^3\sigma hg + \frac{2^4R_2^4\zeta^3\pi}{gh^2}\right) + 20\gamma R_2^2\left(\pi^2 + \frac{4R_2^2\zeta^2}{h^2}\right) + 20\gamma R_2^2\delta^2 \\
& - 30r^2\gamma^2R_2\left(\pi\delta + \frac{2R_2\zeta\sigma}{h}\right) + 20\gamma^2 r\left(\sigma^2\pi g + \frac{2\delta^2\sigma R_2}{h}\right. \\
& \left. + 2\pi^2\zeta R_2 + \frac{2^3\zeta^2\delta R_2^3}{gh^2}\right) + 20\gamma r R_2\left(\pi g + \frac{2R_2\sigma}{h} + 2\zeta R_2\right) \\
& - \frac{40\gamma^2R_2^3\pi\sigma\delta\zeta}{h} - \frac{40\zeta R_2^3}{h}(\pi + \sigma + \delta) - 30\gamma R_2 hg\pi(\sigma + \pi) \\
& - \frac{60}{h}R_2^2 g\sigma(\delta + \sigma) - \frac{240\gamma\delta\zeta R_2^4}{gh^2} + 80\gamma R_2^2\pi\delta + 80r\gamma R_2\left(\sigma g + \frac{2R_2\delta}{h}\right) \\
& \pm 5\sqrt{\gamma}\left[4rR_2\left(g + \frac{2R_2}{h}\right) - 2\gamma R_2\left(\frac{\sigma^3 2R_2 g}{h} + \pi^3 gh + \frac{2^4R_2^4\zeta^3}{gh^2}\right)\right. \\
& - 2R_2\left(\frac{2R_2\delta g}{h} + \frac{8R_2^3\zeta}{gh^2} + \sigma gh\right) - 30r^2\gamma\delta R_2 - 30r^2\gamma R_2\left(\pi + \frac{2R_2\zeta}{h}\right) \\
& + 4R_2\left(\sigma^2 g + \frac{2\delta^2R_2}{h} + \frac{8R_2^3\zeta^2}{gh^2}\right) - \frac{8\zeta R_2^3}{h} - \frac{8\gamma R_2^3\zeta}{h}(\pi\sigma + \pi\delta + \sigma\delta)
\end{aligned}$$

$$- 6\pi R_2 hg(\gamma\pi\sigma + 1) - 12R_2{}^2\frac{\sigma g}{h}(\sigma\delta\gamma + 1) - \frac{48\gamma\delta^2\zeta R_2{}^4}{gh^2}$$

$$+ 8\delta\pi\gamma R_2{}^2(\pi + \delta) + 8R_2{}^2(\pi + \delta) + \frac{32\gamma\sigma\zeta^2 R_2{}^4}{h^2}$$

$$+ 16r\gamma R_2\left(2\pi\zeta R_2 + \pi\sigma g + \frac{2\sigma\delta R_2}{h}\right)\Bigg],$$

$$A_5, B_5 = 2R_2\left(gh^2 + \frac{2g^2R_2}{h} + \frac{2^3R_2{}^3}{gh^2}\right) + 10\gamma^2R_2\left(\pi^4gh^2 + \frac{2\sigma^4g^2R_2}{h} + \frac{2^3R_2{}^3\delta^4}{gh^2}\right)$$

$$+ 20\gamma R_2\left(\pi^2gh^2 + \frac{2R_2\sigma^2g^2}{h} + \frac{2^3R_2{}^3\delta^2}{gh^2}\right) - \frac{40R_2{}^3}{h} - \frac{40\gamma^2R_2{}^3\delta^3}{h}(\sigma + \pi)$$

$$- \frac{320\gamma^2R_2{}^6\zeta^3}{gh^3}(\sigma + \delta) - \frac{40R_2{}^3\zeta\gamma^2}{h}(\sigma^3g + \pi^3h)$$

$$- \frac{40\gamma R_2{}^3}{h}(\sigma\pi + \pi\zeta g + \delta\zeta h) - \frac{120\gamma R_2{}^3}{h}(\sigma\delta + \pi\delta + \zeta\sigma g + \pi\zeta h + \delta^2)$$

$$- \frac{120\gamma^2R_2{}^3\pi}{h}(\sigma\delta^2 + \zeta\sigma^2g + \delta\pi\zeta h) + 20R_2{}^2g(1 + \sigma^2\gamma + \pi^2\gamma)$$

$$+ \frac{80\gamma R_2{}^4\zeta^2}{h}\left(1 + \frac{2R_2}{gh^2}\right) + 20\gamma^2R_2{}^2\left(\sigma^2\pi^2g + \frac{\pi^2\zeta^2\cdot 2^2R_2{}^2}{h} + \frac{2^3R_2{}^3\zeta^2\delta^2}{gh^2}\right)$$

$$+ 40g\gamma R_2{}^2\delta(\pi + \sigma) + \frac{160R_2{}^4\gamma\zeta}{h^2}(\delta + \sigma) + 80\gamma\pi\sigma R_2{}^2g - 10\gamma^2r^3R_2$$

$$- 10\gamma^2r^3R_2\left(\pi + \frac{2R_2\zeta}{h}\right) + 10\gamma r^2gR_2(1 + \sigma^2)$$

$$+ \frac{20\gamma r^2R_2{}^2}{h}\left(1 + \delta^2 + \frac{2^2R_2{}^2\zeta^2}{gh}\right) + 20\gamma^2r^2R_2\left(\pi\sigma g + \frac{2R_2\delta\sigma}{h} + 2R_2\pi\zeta\right)$$

$$- 10r\gamma^2R_2\left(\frac{2R_2\sigma^3g}{h} + \pi^3gh + \frac{2^4R_2{}^4\zeta^3}{gh^2}\right) - 10r\gamma R_2\left(\frac{2R_2\delta g}{h}\right.$$

$$\left.+ \frac{2^3R_2{}^3\zeta}{gh^2} + \sigma gh\right) - \frac{40r\gamma^2R_2{}^3\zeta}{h}(\pi\sigma + \pi\delta + \sigma\delta) - \frac{40r\gamma\zeta R_2{}^3}{h}$$

$$- 30\gamma rR_2g\left(\frac{2R_2\sigma}{h} + \pi h\right) - 30r\gamma^2R_2\left(\frac{2R_2\delta\sigma^2g}{h} + \frac{2^3R_2{}^3\zeta\delta^2}{gh^2} + \pi^2\sigma gh\right)$$

$$+ 40r\gamma R_2{}^2(\pi + \delta) + 40r\gamma^2R_2{}^2\left(\pi\delta^2 + \frac{2^2\sigma\zeta^2R_2{}^2}{h^2} + \delta\pi^2\right)$$

$$\pm\sqrt{\gamma}\left\{\gamma^2\left(2\pi^5gh^2R_2 + \frac{2^8\zeta^5R_2{}^8}{g^2h^4} + \frac{2^2\sigma^5R_2{}^2g^2}{h} + \frac{2^4\delta^5R_2{}^4}{gh^2} + r^5\right)\right.$$

$$+ 10R_2\left(\pi gh^2 + \frac{2\sigma g^2R_2}{h} + \frac{2^3\delta R_2{}^3}{gh^2}\right) + 20\gamma R_2\left(\pi^3gh^2 + \frac{2\sigma^3g^3R_2}{h}\right.$$

$$\left.+ \frac{2^3\delta^3R_2{}^3}{gh^2}\right) - 40\gamma^2R_2{}^3\left(\frac{\sigma\delta^3\pi}{h} + \frac{\sigma\delta\zeta^3 2^3R_2{}^3}{gh^3} + \frac{\pi\sigma^3\zeta g}{h} + \delta\pi^3\zeta\right)$$

$$- \frac{40\gamma R_2{}^3}{h}\left(\delta^3 + \frac{2^3R_2{}^3\zeta^3}{gh^2}\right) - \frac{40R_2{}^3}{h}(\sigma + \pi + \zeta g + \zeta h) - \frac{120\delta R_2{}^3}{h}$$

$$- \frac{120\gamma R_2{}^3\pi}{h}\left(\sigma\delta + \zeta\sigma g + \delta\zeta h + \frac{\sigma\delta^2}{\pi} + \delta^2 + \frac{\zeta\sigma^2g}{\pi} + \zeta\pi h\right)$$

$$+ 20\gamma^2R_2^2\left(\sigma^2\delta\pi^2 g + 2^2R_2^2\cdot\frac{\sigma^2\zeta\delta^2}{h^2} + 2^2R_2^2\cdot\frac{\pi^2\sigma\zeta^2}{h} + 2^3R_2^3\cdot\frac{\delta^2\pi\zeta^2}{gh^2}\right)$$
$$+ 20R_2^2\left(\delta g + \frac{2^2R_2^2\zeta}{h^2}\right) + 20\gamma R_2^2 g\delta(\sigma^2+\pi^2) + \frac{80R_2^4\zeta\gamma}{h^2}(\sigma^2+\delta^2)$$
$$+ \frac{80\gamma R_2^4\zeta}{h}\left(\sigma + \frac{2\pi\zeta R_2}{gh}\right) + 40R_2^2 g(\pi+\sigma) + 80\sigma\delta R_2^2\left(\pi g + \frac{2^2R_2^2\zeta}{h^2}\right)$$
$$- 10\gamma r^3R_2\left(1 + \frac{2R_2\sigma\zeta}{h} + \pi\delta\right) + 10\gamma^2r^2R_2\pi g(1+\sigma^2)$$
$$+ \frac{20\gamma r^2R_2^2}{h}\left(\sigma\delta^2 + \pi^2\zeta h + \sigma + \zeta h + \frac{2^2\zeta^2\delta R_2^2}{g\cdot h}\right)$$
$$+ 20r^2\gamma R_2\left(\sigma g + \frac{2\delta R_2}{h}\right) - 10rgR_2\left(\frac{2R_2}{h} + h\right) + 20rR_2^2$$
$$- 10r\gamma^2R_2\left(\frac{2R_2\sigma^3\delta g}{h} + \frac{2^3R_2^3\delta^3\zeta}{gh^2} + \pi^3\sigma hg + \frac{2^4R_2^4\zeta^3\pi}{gh^2}\right)$$
$$+ 20r\gamma^2R_2^2\left(\frac{2^2R_2^2\sigma^2\zeta^2}{h^2} + \delta^2\pi^2\right) - \frac{40r\gamma R_2^3\zeta}{h}(\pi\sigma\delta\gamma + \pi + \delta + \sigma)$$
$$+ 20\gamma rR_2^2\left(1 + \pi^2\delta^2 + \frac{2^2R_2^2\zeta^2}{h^2}\right) - 30r\gamma R_2 g\left(\frac{2R_2\sigma^2}{h} + \pi^2 h\right)$$
$$+ 80r\gamma\pi\delta R_2^2\Bigg\}.$$

Regarding the sufficiency of the derived conditions we first decide whether $f(x)$ can be reducible under our conditions obtained so far. As ρ_2, ρ_4, ρ_6 each satisfy an irreducible quintic, $\sqrt{\gamma}$ an irreducible quadratic, $\sqrt{\gamma}$ cannot be expressed rationally by ρ_2, ρ_4 or ρ_6, and hence no root can be rational, for we may write $\omega^j\rho_2 + \omega^k\rho_4 + \omega^{-k}\rho_6$ in the form $\frac{\omega^j\rho_6^2}{h} + \frac{\omega^k 2R_2}{\rho_6} + \omega^{-k}\rho_6$. If the last expression were rational, we must have necessarily that ρ_6 is a rational quantity. It follows then, that no root is here rationally known and $f(x)$ has no rational linear factor. The same theory holds for the sum of any two, three or four roots, as ω is rationally known and we have here again, that the sum of any two, three, four or five roots must be irrational except when it is zero which happens only for

$$i\sum_1^5 x_i \quad \text{and} \quad i\sum_1^5 x_{i+5}.$$

Looking into the above A_i and B_i, we see that they contain as only the irrationality $\sqrt{\gamma}$. This irrationality drops out in A_1 and B_1 if

(1) $$r = 0,$$

in A_2 and B_2 if

(2) $$\pi + \delta + \frac{\zeta 2R_2}{h} = 0,$$

and hence in A_3 and B_3 if

(3) $$\left.\begin{array}{r}\pi g(1+\gamma\sigma^2) + 2\sigma g - \dfrac{(\pi+\delta)^3 h\zeta\delta\gamma}{g} \\ -\dfrac{\pi+\delta}{\zeta}[\sigma + \zeta h + \gamma\delta^2\sigma + \gamma\pi^2\zeta h + 2\delta]\end{array}\right\} = 0.$$

in A_4 and B_4 if, in addition to (1) and (2),

$$
\begin{aligned}
& + \frac{\pi+\delta}{\zeta}[\gamma\sigma^3 g + \delta g - 2\delta^2 + 3\sigma g(1+\sigma\delta\gamma)] \\
& + 2\sigma^2 g - \pi^3 hg\gamma - \sigma hg - 3\pi hg(1+\gamma\pi\sigma) \\
(4) \qquad & + \frac{(\pi+\delta)^3 h}{\zeta^2 g}[\gamma\zeta^2 + 3\gamma\delta^2 - 2\gamma\sigma g\zeta + 1 - 2\zeta] \\
& + \frac{(\pi+\delta)^2 h}{\zeta}[+1 - \gamma(\pi\sigma + \sigma\delta + 3\pi\delta)].
\end{aligned}
$$

Finally, assuming the truth of (1) and (2), A_5 and B_5 are rationally known, if

$$
\begin{aligned}
& \pi^5 gh^2\gamma^2 + 5\pi gh^2 + 10\pi^3 gh^2\gamma - 5\frac{\pi+\delta}{\zeta}\left[\frac{\sigma^5 g^2}{5}\gamma^2 + \sigma g^2 + 2\gamma\sigma^3 g^2\right. \\
& \left. + \delta gh(\gamma^2\sigma^2\pi^2 + 1 + \gamma\sigma^2 + \gamma\pi^2 + 4\sigma\pi) + 2gh(\pi+\sigma)\right] \\
& - 5\frac{(\pi+\delta)^2 h}{\zeta^2}\left[\gamma^2\sigma\delta^3\pi + \gamma^2\sigma^3\pi\zeta g + \delta\pi^3\zeta h\gamma^2 + \gamma\delta^3 + \zeta\right. \\
(5) \qquad & \left. + \frac{1}{h}(\sigma + \pi + \zeta g) + 3\delta + \frac{3\gamma\pi}{h}\left(\delta\sigma + \zeta\sigma g + \frac{\sigma\delta^2}{\pi} + \delta^2 + \frac{\zeta\sigma^2 g}{\pi}\right)\right] \\
& - 5\frac{(\pi+\delta)^3 h}{g\zeta^3}\left[+3\gamma\pi\zeta^2 g + \frac{\delta^5}{5}\gamma^2 + \delta + 2\gamma\delta^3 + \gamma^2\sigma^2\zeta\delta^2 g\right. \\
& \left. + \gamma^2 g\pi^2\sigma\zeta^2 h + \zeta g + \zeta g\gamma(\sigma^2+\delta^2) + \gamma\zeta\sigma gh + 4\sigma\delta\zeta g\right] \\
& + 5\frac{(\pi+\delta)^4}{g\zeta^2}h^2\pi\gamma(\delta^2\gamma+1) + 5\frac{(\pi+\delta)^5 h^2}{g\zeta^2}\gamma(1+\gamma\sigma\delta) - \frac{(\pi+\delta)^7}{g^2\zeta^2}h^3\gamma^2.
\end{aligned}
$$

Hence, to assure the irreducibility of $f(x)$, we must have: *Not all of the above five equations must be satisfied.*

If this condition is observed in addition to the conditions already found, then $f(x)$ is irreducible, its group being therefore transitive, must be imprimitive with two as well as five systems of intransitivity, for $f(x)$ contains the five rational factors $(x - x_i)(x - x_{i+5})$, if we adjoin ρ_2, ρ_4 or ρ_6 to the given rational domain; and adjoining $\sqrt{\gamma}$ then the above functions A_i, B_i show that $f(x)$ splits up into the two rational factors

$$\prod_{i=1}^{i=5}(x - x_i) \quad \text{and} \quad \prod_{i=1}^{i=5}(x - x_{i+5}).$$

It follows then that the Galois group of $f(x) = 0$ is a subgroup of G_{240} and it can only be G_{10} under our derived conditions, as σ_1 and σ_2 are impossible substitutions in the group of $f(x) = 0$. σ_1 changes $\rho_2{}^5 \neq 0$ to $\rho_8{}^5 = 0$ and σ_2 applied to $\rho_6{}^5$ produces $\rho_8{}^5 = 0$, so that under the conditions derived $f(x) = 0$ is an irreducible Abelian equation.

Type VIII. If ω, π, σ, δ, ζ, g, h, γ, r, R_2 are quantities in our given rational domain and if $\sqrt{\gamma}$, $\sqrt[5]{2gh^2R_2}$ are not in our given domain, and if r, π, ζ, δ, σ are not all at the same time zero and if the identity on page 157 *exists and if not all five equations*

on pages 160–161 *are satisfied, then* $f(x) = 0$ *of page* 158 *with the roots on page* 158 *is an irreducible Abelian equation.*

An example may be furnished by taking $r = \zeta = \sigma = \delta = 0$.

Here our identity reduces to

$$\frac{1}{2^5\gamma^2} = 2gh^2R_2\pi.$$

Assuming then

$$g = h = R_2 = \pi = 1,$$

we have

$$\gamma^2 = \frac{1}{2^6}; \quad \text{hence} \quad \gamma = \frac{1}{2^3}.$$

$\sqrt{2}$ and $\sqrt[5]{2}$ cannot be in our rational domain, as $\sqrt{\gamma}$ as well as $\sqrt[5]{2gh^2R_2}$ are irrational. Further equation (2) on page 160 is not satisfied.

$$\left.\begin{aligned} f(x) = x^{10} - 20x^8 - 60x^7 + 65x^6 + \frac{8971}{2^4}x^5 + \frac{9075}{2^3}x^4 \\ + \frac{51755}{2^6}x^3 + \frac{65925}{2^7}x^2 + \frac{96475}{2^9}x + \frac{433563}{2^{13}} \end{aligned}\right\} = 0.$$

The roots are:

$$x_2, x_7 = \omega^4\sqrt[5]{4} + \omega^2\sqrt[5]{2} + \omega^3\sqrt[5]{16} \pm \frac{\omega^2\sqrt[5]{2}}{2\sqrt{2}},$$

$$x_1, x_6 = \sqrt[5]{4} + \sqrt[5]{2} + \sqrt[5]{16} \pm \frac{\sqrt[5]{2}}{2\sqrt{2}},$$

$$\cdot \quad \cdot \quad \cdot \quad \text{etc.} \quad \cdot \quad \cdot \quad \cdot$$

The given rational domain must contain ω.

B. II. (3) $\rho_i \neq 0 \quad (i = 2, 4, 6, 8)$.

There is no irreducible Abelian equation of the proposed type. The proof, which is here omitted for brevity, is identical to the proof given under A. II. (3) if we replace there

$$\rho_1, \quad \rho_2, \quad \rho_3, \quad \rho_4, \quad x_i,$$

respectively by

$$\rho_6, \quad \rho_8, \quad \rho_2, \quad \rho_4, \quad \varphi_i.$$

C. CONCLUSION.

There are eight types of irreducible Abelian equations of the tenth degree. Types I to IV are quintics in x^2. *There is no such equation in the natural rational domain* $R(1)$, *for in all types* $\sqrt{5}$ *must be a quantity of the given domain. Six types are possible, only if* ω *belongs to the rational domain.*

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